USEMO
Ersatz Math Olympiad

Language: **English**

Day: **1**

Saturday, October 30, 2021

Problem 1. Let n be a fixed positive integer and consider an $n \times n$ grid of real numbers. Determine the greatest possible number of cells c in the grid such that the entry in c is both strictly *greater* than the average of c's column and strictly *less* than the average of c's row.

Problem 2. Find all integers $n \geq 1$ such that $2^n - 1$ has exactly n positive integer divisors.

Problem 3. Let $A_1C_2B_1A_2C_1B_2$ be an equilateral hexagon. Let O_1 and H_1 denote the circumcenter and orthocenter of $\triangle A_1B_1C_1$, and let O_2 and H_2 denote the circumcenter and orthocenter of $\triangle A_2B_2C_2$. Suppose that $O_1 \neq O_2$ and $H_1 \neq H_2$. Prove that the lines O_1O_2 and H_1H_2 are either parallel or coincide.

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.

Sunday, October 31, 2021

Problem 4. Let ABC be a triangle with circumcircle ω, and let X be the reflection of A in B. Line CX meets ω again at D. Lines BD and AC meet at E, and lines AD and BC meet at F. Let M and N denote the midpoints of AB and AC.

Can line EF share a point with the circumcircle of triangle AMN?

Problem 5. Given a polynomial $p(x)$ with real coefficients, we denote by $S(p)$ the sum of the squares of its coefficients. For example, $S(20x + 21) = 20^2 + 21^2 = 841$.

Prove that if $f(x)$, $g(x)$, and $h(x)$ are polynomials with real coefficients satisfying the identity $f(x) \cdot g(x) = h(x)^2$, then

$$S(f) \cdot S(g) \geq S(h)^2.$$

Problem 6. A *bagel* is a loop of $2a + 2b + 4$ unit squares which can be obtained by cutting a concentric $a \times b$ hole out of an $(a + 2) \times (b + 2)$ rectangle, for some positive integers a and b. (The side of length a of the hole is parallel to the side of length $a + 2$ of the rectangle.)

Consider an infinite grid of unit square cells. For each even integer $n \geq 8$, a *bakery of order* n is a finite set of cells S such that, for every n-cell bagel B in the grid, there exists a congruent copy of B all of whose cells are in S. (The copy can be translated and rotated.) We denote by $f(n)$ the smallest possible number of cells in a bakery of order n.

Find a real number α such that, for all sufficiently large even integers $n \geq 8$, we have

$$\frac{1}{100} < \frac{f(n)}{n^\alpha} < 100.$$

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.