USEMO

Ersatz math olympiad

Language: **English**

Day: **1**

Saturday, October 24, 2020

Problem 1. Which positive integers can be written in the form

$$\frac{\operatorname{lcm}(x, y) + \operatorname{lcm}(y, z)}{\operatorname{lcm}(x, z)}$$

for positive integers x, y, z?

Problem 2. Calvin and Hobbes play a game. First, Hobbes picks a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, 2020\}$, known to both players. Then, Calvin and Hobbes take turns choosing a number from $\{1, 2, \ldots, 2020\}$ which is not already chosen, with Calvin going first, until all numbers are taken (i.e., each player has 1010 numbers). Calvin wins if he has chosen all the elements of some member of $\mathcal{F}$, otherwise Hobbes wins. What is the largest possible size of a family $\mathcal{F}$ that Hobbes could pick while still having a winning strategy?

Problem 3. Let ABC be an acute triangle with circumcenter O and orthocenter H. Let Γ denote the circumcircle of triangle ABC, and N the midpoint of $\overline{OH}$. The tangents to Γ at B and C, and the line through H perpendicular to line AN, determine a triangle whose circumcircle we denote by ω_A. Define ω_B and ω_C similarly.

Prove that the common chords of ω_A, ω_B, and ω_C are concurrent on line OH.

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.

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Language: **English**

Day: **2**

Sunday, October 25, 2020

Problem 4. A function f from the set of positive real numbers to itself satisfies

$$f(x + f(y) + xy) = xf(y) + f(x + y)$$

for all positive real numbers x and y. Prove that $f(x) = x$ for all positive real numbers x.

Problem 5. The sides of a convex 200-gon $A_1A_2 \dots A_{200}$ are colored red and blue in an alternating fashion. Suppose the extensions of the red sides determine a regular 100-gon, as do the extensions of the blue sides.

Prove that the 50 diagonals $\overline{A_1A_{101}}$, $\overline{A_3A_{103}}$, $\dots$, $\overline{A_{99}A_{199}}$ are concurrent.

Problem 6. Prove that for every odd integer $n > 1$, there exist integers $a, b > 0$ such that, if we let $Q(x) = (x + a)^2 + b$, then the following conditions hold:

- we have $\gcd(a, n) = \gcd(b, n) = 1$;
- the number $Q(0)$ is divisible by n; and
- the numbers $Q(1)$, $Q(2)$, $Q(3)$, $\dots$ each have a prime factor not dividing n.

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.