USEMO

Ersatz math olympiad

Language: **English**

Day: **1**

Saturday, May 23, 2020

Problem 1. Let $ABCD$ be a cyclic quadrilateral. A circle centered at O passes through B and D and meets lines BA and BC again at points E and F (distinct from A, B, C). Let H denote the orthocenter of triangle DEF. Prove that if lines AC, DO, EF are concurrent, then triangles ABC and EHF are similar.

Problem 2. Let $\mathbb{Z}[x]$ denote the set of single-variable polynomials in x with integer coefficients. Find all functions $\theta\colon \mathbb{Z}[x] \to \mathbb{Z}[x]$ (i.e. functions taking polynomials to polynomials) such that

- for any polynomials $p, q \in \mathbb{Z}[x]$, $\theta(p+q) = \theta(p) + \theta(q)$;
- for any polynomial $p \in \mathbb{Z}[x]$, p has an integer root if and only if $\theta(p)$ does.

Problem 3. Consider an infinite grid $\mathcal{G}$ of unit square cells. A *chessboard polygon* is a simple polygon (i.e. not self-intersecting) whose sides lie along the gridlines of $\mathcal{G}$.

Nikolai chooses a chessboard polygon F and challenges you to paint some cells of $\mathcal{G}$ green, such that any chessboard polygon congruent to F has at least 1 green cell but at most 2020 green cells. Can Nikolai choose F to make your job impossible?

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.

Sunday, May 24, 2020

Problem 4. Prove that for any prime p, there exists a positive integer n such that

$$1^n + 2^{n-1} + 3^{n-2} + \cdots + n^1 \equiv 2020 \pmod{p}.$$

Problem 5. Let $\mathcal{P}$ be a regular polygon, and let $\mathcal{V}$ be the set of its vertices. Each point in $\mathcal{V}$ is colored red, white, or blue. A subset of $\mathcal{V}$ is *patriotic* if it contains an equal number of points of each color, and a side of $\mathcal{P}$ is *dazzling* if its endpoints are of different colors.

Suppose that $\mathcal{V}$ is patriotic and the number of dazzling edges of $\mathcal{P}$ is even. Prove that there exists a line, not passing through any point of $\mathcal{V}$, dividing $\mathcal{V}$ into two nonempty patriotic subsets.

Problem 6. Let ABC be an acute scalene triangle with circumcenter O and altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Let X, Y, Z be the midpoints of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Lines AD and YZ intersect at P, lines BE and ZX intersect at Q, and lines CF and XY intersect at R.

Suppose that lines YZ and BC intersect at A', and lines QR and EF intersect at D'. Prove that the perpendiculars from A, B, C, O to the lines QR, RP, PQ, $A'D'$, respectively, are concurrent.

Time limit: 4 hours and 30 minutes.
Each problem is worth seven points.