

# USAMO 2026 Solution Notes

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This is a compilation of solutions for the 2026 USAMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the “official” solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered “standard”, then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like “let  $\mathbb{R}$  denote the set of real numbers” are typically omitted entirely.

Corrections and comments are welcome!

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## §0 Problems

1. Fix an integer  $n \geq 2$ . For which real numbers  $x$  is

$$\lfloor nx \rfloor - \sum_{k=1}^n \frac{\lfloor kx \rfloor}{k}$$

maximal, and what is the maximal value that this expression can take?

2. Annie is playing a game where she starts with a row of positive integers, written on a blackboard, each of which is a power of 2. On each turn, she can erase two adjacent numbers and replace them with a power of 2 that is greater than either of the erased numbers. This shortens the row of numbers, and she continues to take turns until only one number remains. Annie wins the game if the final remaining number is less than 4 times the sum of the original numbers. Is it always possible for Annie to win, regardless of the starting row of numbers?
3. Let  $ABC$  be an acute scalene triangle with no angle equal to  $60^\circ$ . Let  $\omega$  be the circumcircle of  $ABC$ . Let  $\Delta_B$  be the equilateral triangle with three vertices on  $\omega$ , one of which is  $B$ . Let  $\ell_B$  be the line through the two vertices of  $\Delta_B$  other than  $B$ . Let  $\Delta_C$  and  $\ell_C$  be defined analogously. Let  $Y$  be the intersection of  $AC$  and  $\ell_B$ , and let  $Z$  be the intersection of  $AB$  and  $\ell_C$ .

Let  $N$  be the midpoint of minor arc  $BC$  on  $\omega$ . Let  $\mathcal{R}$  be the triangle formed by  $\ell_B$ ,  $\ell_C$ , and the tangent to  $\omega$  through  $N$ . Prove that the circumcircle of  $AYZ$  and the incircle of  $\mathcal{R}$  are tangent.

4. A positive integer  $n$  is called *solitary* if, for any non-negative integers  $a$  and  $b$  such that  $a + b = n$ , either  $a$  or  $b$  contains the digit “1”. Determine, with proof, the number of solitary integers less than  $10^{2026}$ .
5. Let  $ABC$  be a triangle. Points  $D$ ,  $E$ , and  $F$  lie on sides  $BC$ ,  $CA$ , and  $AB$ , respectively, such that

$$\angle AFE = \angle BDF = \angle CED.$$

Let  $O_A$ ,  $O_B$ , and  $O_C$  be the circumcenters of triangles  $AFE$ ,  $BDF$ , and  $CED$ , respectively. Let  $M$ ,  $N$ , and  $O$  be the circumcenters of triangles  $ABC$ ,  $DEF$ , and  $O_AO_BO_C$ , respectively. Prove that  $OM = ON$ .

6. Let  $a$  and  $b$  be positive integers such that  $\varphi(ab + 1)$  divides  $a^2 + b^2 + 1$ . Prove that  $a$  and  $b$  are Fibonacci numbers.

## §1 Solutions to Day 1

### §1.1 USAMO 2026/1, proposed by Titu Andreescu and Gabriel Dospinescu

 Available online at <https://aops.com/community/p37578108>.

 Video at <https://youtu.be/lf4skne5Ez0>.

#### Problem statement

Fix an integer  $n \geq 2$ . For which real numbers  $x$  is

$$\lfloor nx \rfloor - \sum_{k=1}^n \frac{\lfloor kx \rfloor}{k}$$

maximal, and what is the maximal value that this expression can take?

¶ **Answer.** The answer is  $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$  and is achieved exactly when the fractional part of  $x$  is at least  $1 - 1/n$ .

¶ **Cosmetic rewriting.** Since the function is invariant under shifting  $x \mapsto x + 1$ , we will focus only on  $x = 1 - y$  for  $0 < y \leq 1$ . In that case, the sum in the problem can be rewritten as

$$\begin{aligned} S(y) &:= \lfloor n(1-y) \rfloor - \sum_{k=1}^n \frac{\lfloor k(1-y) \rfloor}{k} \\ &= n - \lceil ny \rceil - \sum_{k=1}^n \left( 1 - \frac{\lceil ky \rceil}{k} \right) \\ &= \left( \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \right) - \lceil ny \rceil + \sum_{k=1}^n \frac{\lceil ky \rceil - 1}{k} \\ &= \left( \frac{1}{2} + \cdots + \frac{1}{n} \right) - \left[ (\lceil ny \rceil - 1) - \sum_{k=1}^n \frac{\lceil ky \rceil - 1}{k} \right]. \end{aligned}$$

Hence, if we let

$$T(y) = (\lceil ny \rceil - 1) - \sum_{k=1}^n \frac{\lceil ky \rceil - 1}{k}$$

be the bracketed term, it suffices to show  $T(y) \geq 0$  with equality only when  $0 < y \leq \frac{1}{n}$ .

¶ **Main proof.** Let  $p := \lceil ny \rceil - 1$ . If  $p = 0$  (hence if  $0 < y \leq 1/n$ ), then  $T(y) = 0$ , as advertised.

Now assume  $p > 0$ , so that  $\frac{p}{n} < y \leq \frac{p+1}{n}$ . We solve the problem once we prove:

**Claim** (Essentially USAMO 1981/5) — For  $1 \leq p < n$ , we have the inequality

$$\sum_{k=1}^n \frac{\left\lceil k \cdot \frac{p+1}{n} \right\rceil - 1}{k} < p.$$

*Proof.* We rewrite the left-hand side as

$$\begin{aligned} \sum_{k=1}^n \frac{\left\lceil k \cdot \frac{p+1}{n} \right\rceil - 1}{k} &= \sum_{k=1}^n \frac{\sum_{1 \leq d < k \cdot \frac{p+1}{n}} 1}{k} \\ &= \sum_{1 \leq d \leq p} \left( \sum_{d \cdot \frac{n}{p+1} < k \leq n} \frac{1}{k} \right) \\ &= \left( \sum_{\frac{1}{p+1}n < k \leq \frac{2}{p+1}n} \frac{1}{k} \right) + 2 \left( \sum_{\frac{2}{p+1}n < k \leq \frac{3}{p+1}n} \frac{1}{k} \right) + 3 \left( \sum_{\frac{3}{p+1}n < k \leq \frac{4}{p+1}n} \frac{1}{k} \right) \\ &\quad + \cdots + p \left( \sum_{\frac{p}{p+1}n < k \leq \frac{p+1}{p+1}n} \frac{1}{k} \right). \end{aligned}$$

Each of the sums above has at most  $\frac{n}{p+1}$  terms in it. We bound each of the fractions by the lower bound for  $k$ , to get a strict upper bound of

$$\frac{n}{p+1} \left( \frac{1}{\frac{1}{p+1}n} + 2 \cdot \frac{1}{\frac{2}{p+1}n} + \cdots + p \cdot \frac{1}{\frac{p}{p+1}n} \right) = p. \quad \square$$

**§1.2 USAMO 2026/2, proposed by Andrew Dittmer**

 Available online at <https://aops.com/community/p37578099>.

**Problem statement**

Annie is playing a game where she starts with a row of positive integers, written on a blackboard, each of which is a power of 2. On each turn, she can erase two adjacent numbers and replace them with a power of 2 that is greater than either of the erased numbers. This shortens the row of numbers, and she continues to take turns until only one number remains. Annie wins the game if the final remaining number is less than 4 times the sum of the original numbers. Is it always possible for Annie to win, regardless of the starting row of numbers?

To be added later.

**§1.3 USAMO 2026/3, proposed by Rogelio Guerrero**

 Available online at <https://aops.com/community/p37578103>.

**Problem statement**

Let  $ABC$  be an acute scalene triangle with no angle equal to  $60^\circ$ . Let  $\omega$  be the circumcircle of  $ABC$ . Let  $\Delta_B$  be the equilateral triangle with three vertices on  $\omega$ , one of which is  $B$ . Let  $\ell_B$  be the line through the two vertices of  $\Delta_B$  other than  $B$ . Let  $\Delta_C$  and  $\ell_C$  be defined analogously. Let  $Y$  be the intersection of  $AC$  and  $\ell_B$ , and let  $Z$  be the intersection of  $AB$  and  $\ell_C$ .

Let  $N$  be the midpoint of minor arc  $BC$  on  $\omega$ . Let  $\mathcal{R}$  be the triangle formed by  $\ell_B$ ,  $\ell_C$ , and the tangent to  $\omega$  through  $N$ . Prove that the circumcircle of  $AYZ$  and the incircle of  $\mathcal{R}$  are tangent.

To be added later.

## §2 Solutions to Day 2

### §2.1 USAMO 2026/4, proposed by Carl Schildkraut

Available online at <https://aops.com/community/p37586238>.

Video at <https://youtu.be/0gbE7VKeYEs>.

#### Problem statement

A positive integer  $n$  is called *solitary* if, for any non-negative integers  $a$  and  $b$  such that  $a + b = n$ , either  $a$  or  $b$  contains the digit “1”. Determine, with proof, the number of solitary integers less than  $10^{2026}$ .

We claim that a number is solitary if and only if all the following hold:

- The digit 1 appears exactly once;
- Every digit (possibly none) to the left of the 1 is 0 or 2;
- Every digit (possibly none) to the right of the 1 is 9.

For example, 202201999999 is solitary.

¶ **Proof all such numbers are solitary.** The basic idea is to use induction as follows: If the last digit of  $n$  is 9, then the last digits of  $a$  and  $b$  sum to 9; hence we can ignore the last digit altogether. Thus, we reduce to the case where the last digit is 1.

We continue the induction in a similar way in this situation. Zero-pad all the numbers to be the same length as  $n$ . Take the leading 2 of  $n$ .

- If either  $a$  or  $b$  has a leading digit 2, we can delete it and continue the induction.
- Otherwise, clearly one of  $a$  or  $b$  must have leading digit 1, as needed.

¶ **Proof that every solitary number is of this form.** Let  $n$  be solitary. In all the diagrams that follow, ellipses denote groups of digits other than 1 (possibly none).

We first reduce to the case where  $n$  has exactly one 1:

- By taking  $b = 0$ , we see there is at least a single 1.
- Suppose  $n$  has an even number of 1’s. The idea is to pair the 1’s using blocks like  $9\dots 9$ , as in the following diagram:

$$\begin{array}{rcccccc} n = & \dots 1 & \dots 1 & \dots 1 & \dots 1 & \dots \\ a = & 0000 & 9993 & 0000 & 9993 & 0000 \\ b = & \dots 0 & \dots 8 & 0000 & \dots 8 & \dots \end{array}$$

- Next, suppose  $n$  has an odd number of 1’s and at least three 1’s. The strategy is similar:

$$\begin{array}{rcccccc} n = & \dots 1 & \dots 1 & \dots 1 & \dots 1 & \dots 1 & \dots \\ a = & 0000 & 9993 & 9993 & 0000 & 9993 & 0000 \\ b = & \dots 1 & \dots 7 & \dots 8 & 0000 & \dots 8 & \dots \end{array}$$

So, assume  $n$  has exactly one 1 appear.

- If a digit  $b \neq 0, 9$  appears to the right of 1, let  $c = b + 1$  to get

$$\begin{array}{rcccc} n = & \dots & 1 & \dots b & \dots \\ a = & 0000 & 0 & 9999 & 0000 \\ b = & \dots & 0 & \dots c & \dots \end{array}$$

If  $b = 0$ , one can instead replace 9999 with 9998 and choose  $c = 2$ .

- Finally, if a digit  $d \neq 0, 2$  appears to the left of 1, let  $e = d - 1$  and use a bunch of 9's:

$$\begin{array}{rcccc} n = & \dots d & \dots & 1 & \dots \\ a = & 0000 & 9999 & 9 & 0000 \\ a = & \dots e & \dots & 2 & \dots \end{array}$$

¶ **Final count.** Zero-pad the number so that it has exactly 2026 digits. If we pick the digit 1 to appear at the  $i$ 'th place from the left, for  $1 \leq i \leq 2026$ , then there are  $2^{i-1}$  ways to pick the first  $i - 1$  digits (either 0 or 2); the rest are all 9. Hence, the answer is

$$\sum_{i=1}^{2026} 2^{i-1} = \boxed{2^{2026} - 1}.$$

## §2.2 USAMO 2026/5, proposed by Serena An

Available online at <https://aops.com/community/p37586259>.

### Problem statement

Let  $ABC$  be a triangle. Points  $D$ ,  $E$ , and  $F$  lie on sides  $BC$ ,  $CA$ , and  $AB$ , respectively, such that

$$\angle AFE = \angle BDF = \angle CED.$$

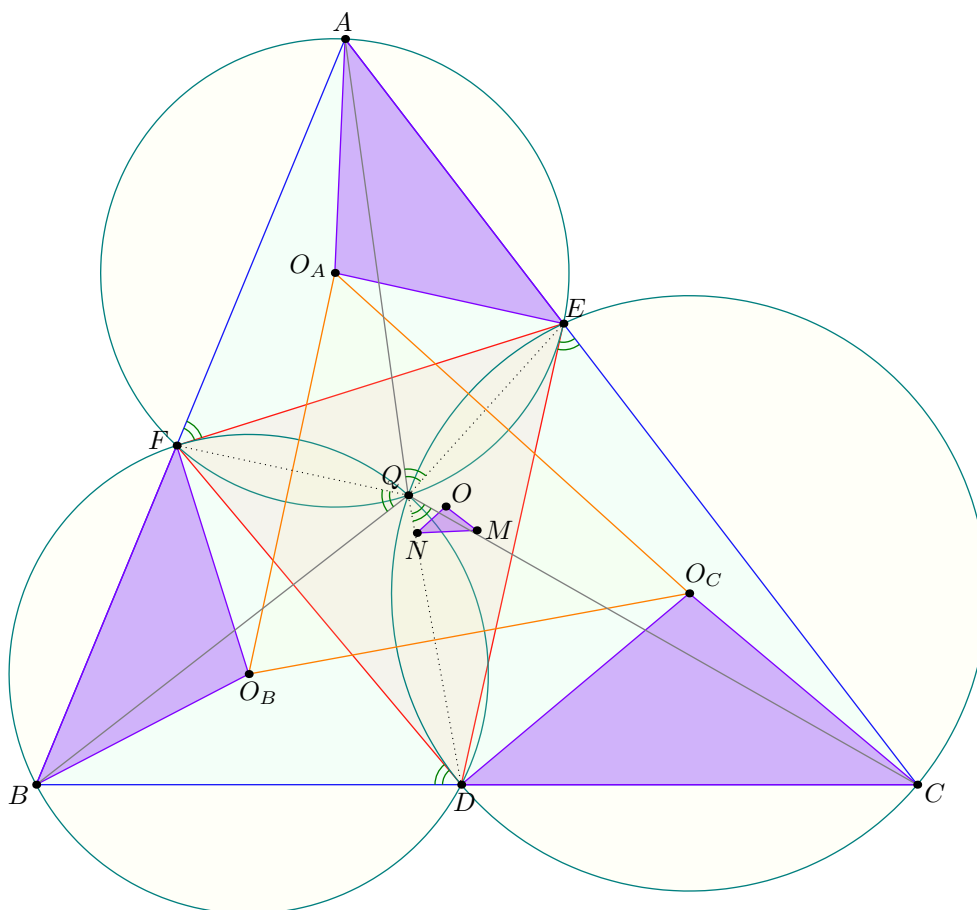
Let  $O_A$ ,  $O_B$ , and  $O_C$  be the circumcenters of triangles  $AFE$ ,  $BDF$ , and  $CED$ , respectively. Let  $M$ ,  $N$ , and  $O$  be the circumcenters of triangles  $ABC$ ,  $DEF$ , and  $O_AO_BO_C$ , respectively. Prove that  $OM = ON$ .

For now, entirely ignore the points  $M$ ,  $N$ ,  $O$ ; we'll deal with them at the end.

By Miquel theorem, the circumcircles of  $AEF$ ,  $BFD$ , and  $CDE$  meet at a point  $Q$ , the most important point in this solution. Define the angles

$$\alpha = \angle AFE = \angle BDF = \angle CED$$

$$\beta = \angle QEA = \angle QEC = \angle QDC = \angle QDB = \angle QFB = \angle QFA.$$



We will make explicit a spiral similarity from  $\triangle ABC$  to  $\triangle DEF$  centered at  $Q$ . It will be best stated in the following form:

**Claim** — We have the spiral similarities

$$\triangle QAE \stackrel{\pm}{\sim} \triangle QBF \stackrel{\pm}{\sim} \triangle QCD.$$

*Proof.* Note that  $\angle AQE = \angle AFE = \alpha$  (marked in green in the figure), while  $\angle QEA = \beta$ .  $\square$

In fact, now we claim  $\triangle O_A O_B O_C$  is similar too.

**Claim** — The same spiral similarities at  $Q$  map

$$\triangle AO_A E \stackrel{\pm}{\sim} \triangle BO_B F \stackrel{\pm}{\sim} \triangle CO_C D.$$

*Proof.* These triangles are isosceles with  $\angle AO_A E = 2\alpha$ .  $\square$

Finally we bring in the points  $M, N, O$  used only for extraction. Since our spiral similarities map  $ABC, DEF$ , and  $O_A O_B O_C$  to each other, and these are the respective circumcenters, we can extend our similarity to

$$\triangle AO_A E \stackrel{\pm}{\sim} \triangle BO_B F \stackrel{\pm}{\sim} \triangle CO_C D \stackrel{\pm}{\sim} \triangle MON.$$

In particular,  $OM = ON$ .

**Remark.** The choice of  $M, N, O$  as the *circumcenters* (as opposed to any other center) of the three triangles  $ABC, DEF, O_A O_B O_C$  is completely superficial and has no effect on the proof. That is, one could have replaced those with any triangle center and the proof would carry over verbatim.

**Remark.** Actually  $\angle QAE = \angle QFE = \angle QFA - \alpha = \beta - \alpha$ , so by symmetry we actually also know

$$\angle QAE = \angle QBF = \angle QCD = \beta - \alpha.$$

So the point  $Q$  is the so-called second Brocard point of the triangle.

## §2.3 USAMO 2026/6, proposed by Raymond Feng and Eric Shen

Available online at <https://aops.com/community/p37586266>.

Video at <https://youtu.be/FJX1iXhpXVQ>.

### Problem statement

Let  $a$  and  $b$  be positive integers such that  $\varphi(ab + 1)$  divides  $a^2 + b^2 + 1$ . Prove that  $a$  and  $b$  are Fibonacci numbers.

We start with the following.

**Claim** — The number  $ab + 1$  must be a prime power.

*Proof.* Note to start that:

- Since Euler phi only gives even values,  $2 \mid a^2 + b^2 + 1$ .
- Hence, exactly one of  $a$  and  $b$  is odd.
- Thus,  $a^2 + b^2 + 1 \equiv 2 \pmod{4}$ .

So  $\nu_2(\varphi(ab + 1)) = 1$ , and hence  $ab + 1$  must have at most one prime factor.  $\square$

Henceforth, let  $ab + 1 = p^e$ . The  $e = 1$  case is easy, and does not even rely on  $ab + 1$  being prime:

**Claim** — The equation  $ab \mid a^2 + b^2 + 1$  is equivalent to  $\{a, b\} = \{F_{2k-1}, F_{2k+1}\}$  for  $k \geq 0$ . Here the Fibonacci sequence  $F_i$  is indexed by  $F_{-1} = F_1 = F_2 = 1$  and  $F_0 = 0$ .

*Proof.* Classical Vieta jumping argument.  $\square$

We now consider  $e \geq 2$ . In that case, we have

$$\begin{aligned} \varphi(p^e) = p^{e-1} \cdot (p-1) \mid a^2 + b^2 + 1 \\ p^e - 1 = ab. \end{aligned}$$

This implies that

$$0 \equiv a^2 + \left(\frac{p^e - 1}{a}\right)^2 + 1 \equiv \frac{a^4 + a^2 + 1}{a^2} = \frac{(a^2 + a + 1)(a^2 - a + 1)}{a^2} \pmod{p^{e-1}}.$$

In particular, this implies  $p = 3$  or  $p \equiv 1 \pmod{3}$ .

- If  $p \equiv 1 \pmod{3}$ , then we get  $3 \mid p - 1 \mid a^2 + b^2 + 1$ , which forces  $a$  and  $b$  to be nonzero modulo 3. However, then  $p^e - 1 = ab \not\equiv 0 \pmod{3}$ , which is impossible.
- Now suppose  $p = 3$ . Since  $x^2 \pm x + 1$  is never divisible by 9, this forces  $e \leq 2$ . Hence, we only have to consider the case  $ab = 8$ . We find there is one additional solution here only:  $(1, 8)$ , and both 1 and 8 are Fibonacci numbers.