USAMO 2021 Solution Notes

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This is a compilation of solutions for the 2021 USAMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

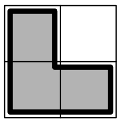
1. Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

2. The Planar National Park is a undirected 3-regular planar graph (i.e. all vertices have degree 3). A visitor walks through the park as follows: she begins at a vertex and starts walking along an edge. When she reaches the other endpoint, she turns left. On the next vertex she turns right, and so on, alternating left and right turns at each vertex. She does this until she gets back to the vertex where she started. What is the largest possible number of times she could have entered any vertex during her walk, over all possible layouts of the park?

3. Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves:
   - If there is an L-shaped tromino region of three cells without stones on the board (see figure; rotations not allowed), you may place a stone in each of those cells.



   - If all cells in a column have a stone, you may remove all stones from that column.
   - If all cells in a row have a stone, you may remove all stones from that row.

   For which n is it possible that, after some non-zero number of moves, the board has no stones?

4. A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

   Given this information, find all possible values for the number of elements of S.

5. Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned}
a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\
a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\
a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\
&\vdots & &\vdots \\
a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1.
\end{aligned}$$

6. Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

§1 Solutions to Day 1

§1.1 USAMO 2021/1, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p21498558.

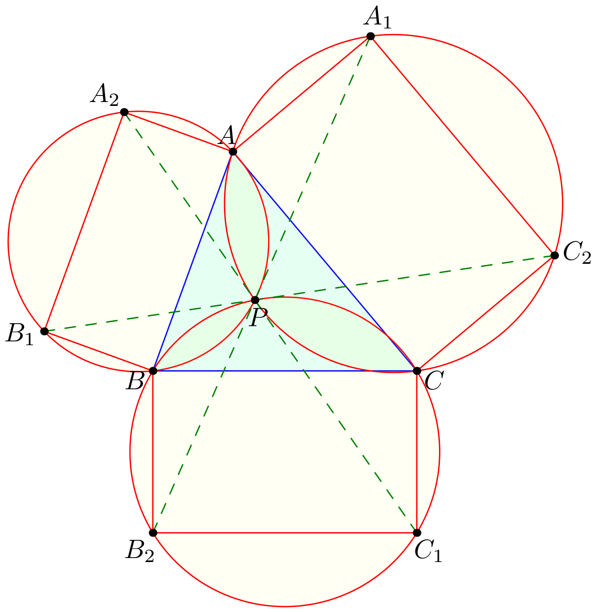
Problem statement

Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

The angle condition implies the circumcircles of the three rectangles concur at a single point P.



Then $\measuredangle CPB_2 = \measuredangle CPA_1 = 90^\circ$, hence P lies on A_1B_2 etc., so we're done.

Remark. As one might guess from the two-sentence solution, the entire difficulty of the problem is getting the characterization of the concurrence point.

§1.2 USAMO 2021/2, proposed by Zoran Sunic

Available online at https://aops.com/community/p21498640.

Problem statement

The Planar National Park is a undirected 3-regular planar graph (i.e. all vertices have degree 3). A visitor walks through the park as follows: she begins at a vertex and starts walking along an edge. When she reaches the other endpoint, she turns left. On the next vertex she turns right, and so on, alternating left and right turns at each vertex. She does this until she gets back to the vertex where she started. What is the largest possible number of times she could have entered any vertex during her walk, over all possible layouts of the park?

The answer is 3.

We consider the trajectory of the visitor as an ordered sequence of *turns*. A turn is defined by specifying a vertex, the incoming edge, and the outgoing edge. Hence there are six possible turns for each vertex.

Claim — Given one turn in the sequence, one can reconstruct the entire sequence of turns.

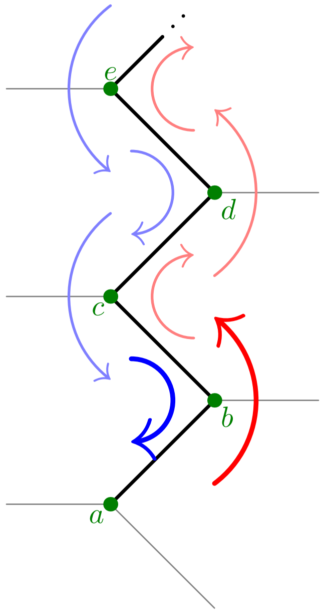
Proof. This is clear from the process's definition: given a turn t, one can compute the turn after it and the turn before it. □

This implies already that the trajectory of the visitor, when extended to an infinite sequence, is totally periodic (not just eventually periodic), because there are finitely many possible turns, so some turn must be repeated. So, any turn appears at most once in the period of the sequence, giving a naïve bound of 6 for the original problem.

However, the following claim improves the bound to 3.

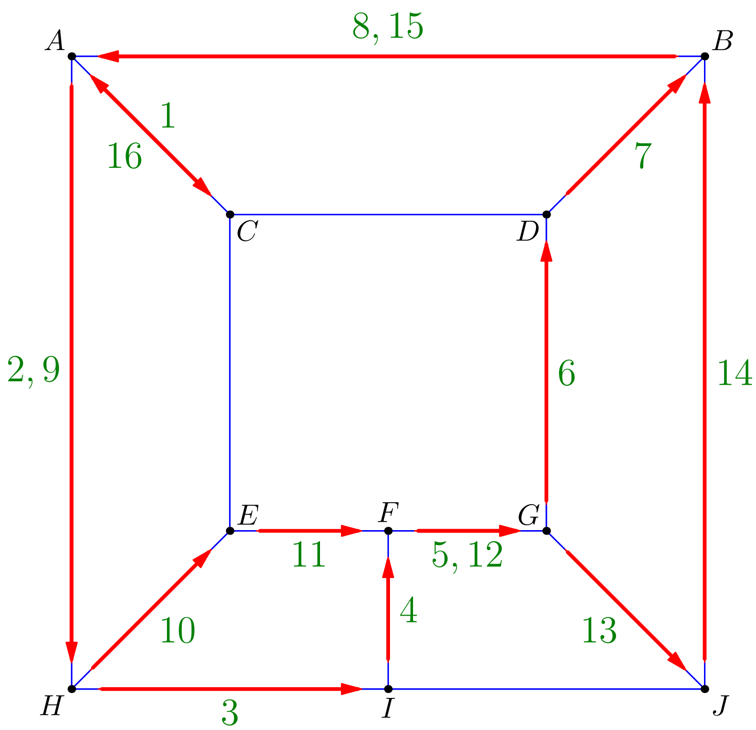
Claim — It is impossible for both of the turns $a \to b \to c$ and $c \to b \to a$ to occur in the same trajectory.

Proof. If so, then extending the path, we get $a \to b \to c \to d \to e \to \cdots$ and $\cdots \to e \to d \to c \to b \to a$, as illustrated below in red and blue respectively.



However, we assumed for contradiction the red and blue paths were part of the same trajectory, yet they clearly never meet. □

It remains to give a construction showing 3 can be achieved. There are many, many valid constructions. One construction due to Danielle Wang is given here, who provided the following motivation: "I was lying in bed and drew the first thing I could think of". The path is $CAHIFGDBAHEFGJBAC$ which visits A three times.



> **Remark.** As the above example shows it is possible to transverse an edge more than once even in the same direction, as in edge AH above.

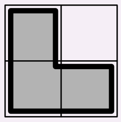
§1.3 USAMO 2021/3, proposed by Alex Zhai, Shaunak Kishore

Available online at https://aops.com/community/p21498538.

Problem statement

Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves:

- If there is an L-shaped tromino region of three cells without stones on the board (see figure; rotations not allowed), you may place a stone in each of those cells.

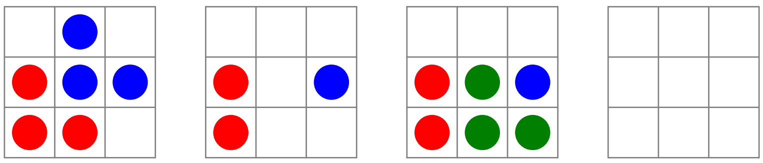


- If all cells in a column have a stone, you may remove all stones from that column.
- If all cells in a row have a stone, you may remove all stones from that row.

For which n is it possible that, after some non-zero number of moves, the board has no stones?

The answer is $3 \mid n$.

Construction: For $n = 3$, the construction is fairly straightforward, shown below.



This can be extended to any $3 \mid n$.

Polynomial-based proof of converse: Assume for contradiction $3 \nmid n$. We will show the task is impossible even if we allow stones to have real weights in our process. A valid elimination corresponds to a polynomial $P \in \mathbb{R}[x, y]$ such that

$$\begin{aligned} \deg_x P &\leq n - 2 \\ \deg_y P &\leq n - 2 \\ (1 + x + y)P(x, y) &\in \left\langle 1 + x + \cdots + x^{n-1},\ 1 + y + \cdots + y^{n-1} \right\rangle. \end{aligned}$$

(Here $\langle \dots \rangle$ is an ideal of $\mathbb{R}[x, y]$.) In particular, if S is the set of nth roots of unity other than 1, we should have

$$(1 + z_1 + z_2)P(z_1, z_2) = 0$$

for any $z_1, z_2 \in S$. Since $3 \nmid n$, it follows that $1 + z_1 + z_2 \neq 0$ always.

So P vanishes on $S \times S$, a contradiction to the bounds on $\deg P$ (by, say, combinatorial nullstellensatz on any nonzero term).

Linear algebraic proof of converse (due to William Wang): Suppose there is a valid sequence of moves. Call r_j the number of operations clearing row j, indexing from bottom-to-top. The idea behind the solution is that we are going to calculate, for each cell, the number of times it is operated on entirely as a function of r_j. For example, a hypothetical illustration with $n = 6$ is partially drawn below, with the number in each cell denoting how many times it was the corner of an L.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 = r_3 & c_4 = r_5 - r_4 & c_5 = r_5 & 0 \\ \vdots & \vdots & r_2 + r_3 - r_5 & r_5 - r_3 & r_4 & 0 \\ \vdots & \vdots & r_1 + r_2 + r_3 - r_4 - r_5 & r_5 - r_2 & r_3 & 0 \\ \vdots & \vdots & r_1 + r_2 + r_4 - r_5 & r_5 - r_1 & r_2 & 0 \\ \vdots & \vdots & r_1 + r_4 - r_5 & r_5 & r_1 & 0 \end{bmatrix}$$

Let $a_{i,j}$ be the expression in (i, j). It will also be helpful to define c_i in the obvious way as well.

Claim — We have $c_n = r_n = 0$, $a_{n-1,j} = r_j$ and $a_{i,n-1} = c_i$.

Proof. The first statement follows since (n, n) may never obtain a stone. The equation $a_{n-1,j} = r_j$ follows as both equal the number of times that cell (n, j) obtains a stone. The final equation is similar. □

Claim — For $1 \le i, j \le n-1$, the following recursion holds:

$$a_{i,j} + a_{i+1,j} + a_{i+1,j-1} = r_j + c_{i+1}.$$

Proof. Focus on cell $(i + 1, j)$. The left-hand side counts the number of times that gains a stone while the right-hand side counts the number of times it loses a stone; they must be equal. □

We can coerce the table above into matrix form now as follows. Define

$$K = \begin{bmatrix} -1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 \end{bmatrix}.$$

Then define a sequence of matrices M_i recursively by $M_{n-1} = \mathrm{id}$, and

$$M_i = \mathrm{id} + KM_{i+1} = \mathrm{id} + K + \dots + K^{n-1-i}.$$

The matrices are chosen so that, by construction,

$$\langle r_1, \dots, r_{n-1} \rangle M_i = \langle a_{i,1}, \dots, a_{i,n-1} \rangle$$

for $i = 1, 2, \dots, n-1$. On the other hand, we can extend the recursion one level deeper and obtain

$$\langle r_1, \dots, r_{n-1} \rangle M_0 = \langle 0, \dots, 0 \rangle .$$

However, the crux of the solution is the following.

Claim — The eigenvalues of K are exactly $-(1+e^{\frac{2\pi ik}{n}})$ for $k = 1, 2, \dots, n-1$.

Proof. The matrix $-(K+\mathrm{id})$ is fairly known to have roots of unity as the coefficients. $\square$

However, we are told that apparently

$$0 = \det M_0 = \det\left(\mathrm{id} + K + K^2 + \cdots + K^{n-1}\right)$$

which means $\det(K^n - \mathrm{id}) = 0$. This can only happen if K^n has eigenvalue 1, meaning that

$$[-(1+\omega)]^n = 1$$

for ω some n^{th} root of unity, not necessarily primitive. This can only happen if $|1+\omega| = 1$, ergo $3 \mid n$.

§2 Solutions to Day 2

§2.1 USAMO 2021/4, proposed by Carl Schildkraut

Available online at https://aops.com/community/p21498580.

Problem statement

A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

The answer is that $|S|$ must be a power of 2 (including 1), or $|S| = 0$ (a trivial case we do not discuss further).

¶ Construction. For any nonnegative integer k, a construction for $|S| = 2^k$ is given by

$$S = \{(p_1 \text{ or } q_1) \times (p_2 \text{ or } q_2) \times \cdots \times (p_k \text{ or } q_k)\}$$

for $2k$ distinct primes $p_1, \ldots, p_k, q_1, \ldots, q_k$.

¶ Converse. The main claim is as follows.

Claim — In any valid set S, for any prime p and $x \in S$, $\nu_p(x) \le 1$.

Proof. Assume for contradiction $e = \nu_p(x) \ge 2$.

- On the one hand, by taking x in the statement, we see $\frac{e}{e+1}$ of the elements of S are divisible by p.
- On the other hand, consider a $y \in S$ such that $\nu_p(y) = 1$ which must exist (say if $\gcd(x,y) = p$). Taking y in the statement, we see $\frac{1}{2}$ of the elements of S are divisible by p.

So $e = 1$, contradiction. □

Now since $|S|$ equals the number of divisors of any element of S, we are done.

§2.2 USAMO 2021/5, proposed by Mohsen Jamaali

Available online at https://aops.com/community/p21498967.

Problem statement

Let $n \ge 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned} a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\ a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\ a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\ &\vdots & &\vdots \\ a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1. \end{aligned}$$

The answer is that the only solution is $(1, 2, 1, 2, \dots, 1, 2)$ which works.

We will prove a_{2k} is a constant sequence, at which point the result is obvious.

¶ First approach (Andrew Gu). Apparently, with indices modulo $2n$, we should have

$$a_{2k} = \frac{1}{a_{2k-2}} + \frac{2}{a_{2k}} + \frac{1}{a_{2k+2}}$$

for every index k (this eliminates all a_{odd}'s). Define

$$m = \min_k a_{2k} \qquad \text{and} \qquad M = \max_k a_{2k}.$$

Look at the indices i and j achieving m and M to respectively get

$$\begin{aligned} m &= \frac{2}{m} + \frac{1}{a_{2i-2}} + \frac{1}{a_{2i+2}} \ge \frac{2}{m} + \frac{1}{M} + \frac{1}{M} = \frac{2}{m} + \frac{2}{M} \\ M &= \frac{2}{M} + \frac{1}{a_{2j-2}} + \frac{1}{a_{2j+2}} \le \frac{2}{M} + \frac{1}{m} + \frac{1}{m} = \frac{2}{m} + \frac{2}{M}. \end{aligned}$$

Together this gives $m \ge M$, so $m = M$. That means a_{2i} is constant as i varies, solving the problem.

¶ Second approach (author's solution). As before, we have

$$a_{2k} = \frac{1}{a_{2k-2}} + \frac{2}{a_{2k}} + \frac{1}{a_{2k+2}}$$

The proof proceeds in three steps.

- Define
$$S = \sum_k a_{2k}, \quad \text{and} \quad T = \sum_k \frac{1}{a_{2k}}.$$
Summing gives $S = 4T$. On the other hand, Cauchy-Schwarz says $S \cdot T \ge n^2$, so $T \ge \frac{1}{2}n$.

- On the other hand,
$$1 = \frac{1}{a_{2k-2}a_{2k}} + \frac{2}{a_{2k}^2} + \frac{1}{a_{2k}a_{2k+2}}$$
Sum this modified statement to obtain
$$n = \sum_k \left(\frac{1}{a_{2k}} + \frac{1}{a_{2k+2}}\right)^2 \overset{\text{QM-AM}}{\geq} \frac{1}{n}\left(\sum_k \frac{1}{a_{2k}} + \frac{1}{a_{2k+2}}\right)^2 = \frac{1}{n}(2T)^2$$
So $T \leq \frac{1}{2}n$.
- Since $T \leq \frac{1}{2}n$ and $T \geq \frac{1}{2}n$, we must have equality everywhere above. This means a_{2k} is a constant sequence.

Remark. The problem is likely intractable over $\mathbb{C}$, in the sense that one gets a high-degree polynomial which almost certainly has many complex roots. So it seems likely that most solutions must involve some sort of inequality, using the fact we are over $\mathbb{R}_{>0}$ instead.

§2.3 USAMO 2021/6, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p21498548.

Problem statement

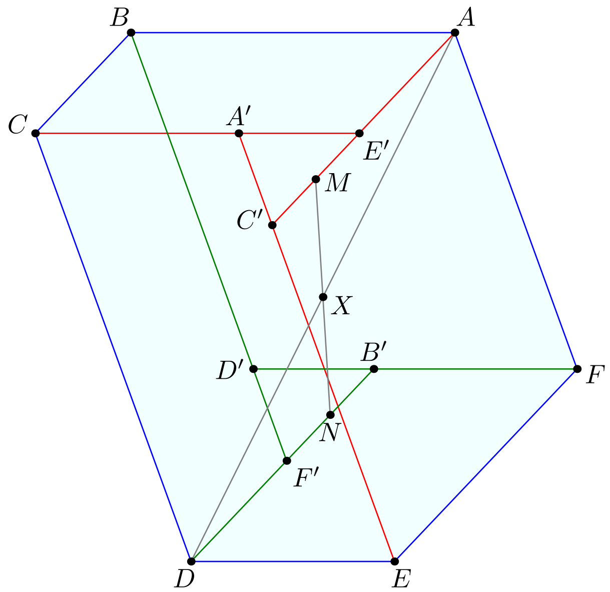
Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

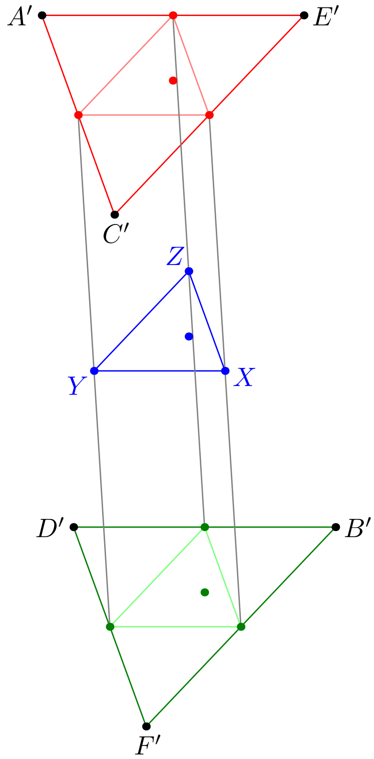
We present two solutions.

¶ Parallelogram solution found by contestants. Note that the following figure is intentionally *not* drawn to scale, to aid legibility. We construct parallelograms $ABCE'$, etc as shown. Note that this gives two congruent triangles $A'C'E'$ and $B'D'F'$. (Assuming that triangle XYZ is non-degenerate, the triangles $A'C'E'$ and $B'D'F'$ will also be non-degenerate.)



Claim — If $AB \cdot DE = BC \cdot EF = CD \cdot FA = k$, then the circumcenters of ACE and $A'C'E'$ coincide.

Proof. The power of A to $(A'C'E')$ is $AE' \cdot AC' = BC \cdot EF = k$; same for C and E. $\square$



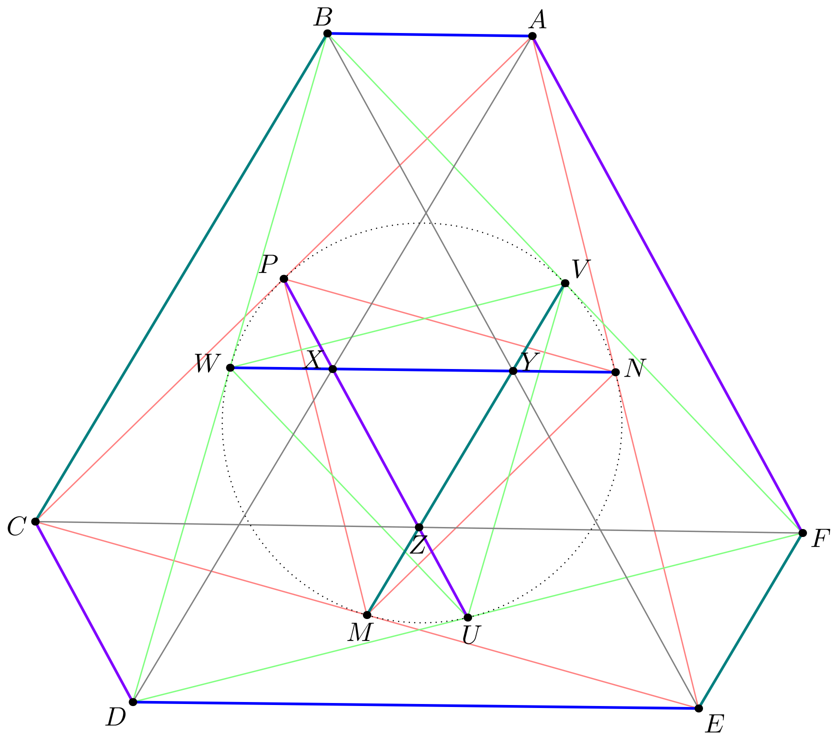
> **Claim** — Triangle XYZ is the vector average of the (congruent) medial triangles of triangles $A'C'E'$ and $B'D'F'$.

Proof. If M and N are the midpoints of $\overline{C'E'}$ and $\overline{B'F'}$, then X is the midpoint of $\overline{MN}$ by vector calculation:

$$\begin{aligned}\frac{\vec{M}+\vec{N}}{2} &= \frac{\frac{\vec{C}'+\vec{E}'}{2}+\frac{\vec{B}'+\vec{F}'}{2}}{2} \\ &= \frac{\vec{C}'+\vec{E}'+\vec{B}'+\vec{F}'}{4} \\ &= \frac{(\vec{A}+\vec{E}-\vec{F})+(\vec{C}+\vec{A}-\vec{B})+(\vec{D}+\vec{F}-\vec{E})+(\vec{B}+\vec{D}-\vec{C})}{4} \\ &= \frac{\vec{A}+\vec{D}}{2} = \vec{X}.\end{aligned}$$ □

Hence the orthocenter of XYZ is the midpoint of the orthocenters of the medial triangles of $A'C'E'$ and $B'D'F'$ — that is, their circumcenters.

¶ **Author's solution.** Call MNP and UVW the medial triangles of ACE and BDF.



Claim — In trapezoid $ABDE$, the perpendicular bisector of $\overline{XY}$ is the same as the perpendicular bisector of the midline $\overline{WN}$.

Proof. This is true for any trapezoid: because $WX = \frac{1}{2}AB = YN$. □

Claim — The points V, W, M, N are cyclic.

Proof. By power of a point from Y, since

$$WY \cdot YN = \frac{1}{2}DE \cdot \frac{1}{2}AB = \frac{1}{2}EF \cdot \frac{1}{2}BC = VY \cdot YM.$$ □

Applying all the cyclic variations of the above two claims, it follows that all six points U, V, W, M, N, P are concyclic, and the center of that circle coincides with the circumcenter of $\triangle XYZ$.

Remark. It is also possible to implement ideas from the first solution here, by showing all six midpoints have equal power to (XYZ).

Claim — The orthocenter of XYZ is the midpoint of the circumcenters of $\triangle ACE$ and $\triangle BDF$.

Proof. Apply complex numbers with the unit circle coinciding with the circumcircle of $NVPWMU$. Then

$$\text{orthocenter}(XYZ) = x + y + z = \frac{a+b+c+d+e+f}{2}$$
$$\text{circumcenter}(ACE) = \text{orthocenter}(MNP)$$

$$\begin{aligned} &= m + n + p = \frac{c+e}{2} + \frac{e+a}{2} + \frac{a+c}{2} = a + c + e \\ \text{circumcenter}(BDF) &= \text{orthocenter}(UVW) \\ &= u + v + w = \frac{d+f}{2} + \frac{f+b}{2} + \frac{b+d}{2} = b + d + f. \end{aligned}$$

$\square$