USAMO 2016 Solution Notes

Evan Chen《陳誼廷》

15 April 2024

This is a compilation of solutions for the 2016 USAMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let $X_1, X_2, \dots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \dots, 99\}$. Find the smallest possible number of elements in S.

2. Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

3. Let ABC be an acute triangle and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines I_BF and I_CE meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

4. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

5. An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overline{MN}$ and $\overline{PQ}$. Denote by ℓ the angle bisector of $\angle MSQ$.

Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

6. Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard. The wizard has $2n$ cards; for each $i = 1, \dots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order. You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is *winnable* if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds. For which values of n and k is the game winnable?

§1 Solutions to Day 1

§1.1 USAMO 2016/1, proposed by Iurie Boreico

Available online at https://aops.com/community/p6213589.

Problem statement

Let $X_1, X_2, \dots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \dots, 99\}$. Find the smallest possible number of elements in S.

Solution with Danielle Wang: the answer is that $|S| \geq 8$.

¶ Proof that $|S| \geq 8$ is necessary. Since we must have $2^{|S|} \geq 100$, we must have $|S| \geq 7$.

To see that $|S| = 8$ is the minimum possible size, consider a chain on the set $S = \{1, 2, \dots, 7\}$ satisfying $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$. Because of these requirements any subset of size 4 or more can only be neighbored by sets of size 2 or less, of which there are $\binom{7}{1} + \binom{7}{2} = 28$ available. Thus, the chain can contain no more than 29 sets of size 4 or more and no more than 28 sets of size 2 or less. Finally, since there are only $\binom{7}{3} = 35$ sets of size 3 available, the total number of sets in such a chain can be at most $29 + 28 + 35 = 92 < 100$, contradiction.

¶ Construction. We will provide an inductive construction for a chain of subsets $X_1, X_2, \dots, X_{2^{n-1}+1}$ of $S = \{1, \dots, n\}$ satisfying $X_i \cap X_{i+1} = \varnothing$ and $X_i \cup X_{i+1} \neq S$ for each $n \geq 4$.

For $S = \{1, 2, 3, 4\}$, the following chain of length $2^3 + 1 = 9$ will work:

$$34 \quad 1 \quad 23 \quad 4 \quad 12 \quad 3 \quad 14 \quad 2 \quad 13 \ .$$

Now, given a chain of subsets of $\{1, 2, \dots, n\}$ the following procedure produces a chain of subsets of $\{1, 2, \dots, n+1\}$:

1. take the original chain, delete any element, and make two copies of this chain, which now has even length;
2. glue the two copies together, joined by $\varnothing$ in between; and then
3. insert the element $n + 1$ into the sets in alternating positions of the chain starting with the first.

For example, the first iteration of this construction gives:

$$\begin{array}{ccccccccc} 345 & 1 & 235 & 4 & 125 & 3 & 145 & 2 & 5 \\ 34 & 15 & 23 & 45 & 12 & 35 & 14 & 25 & \end{array}$$

It can be easily checked that if the original chain satisfies the requirements, then so does the new chain, and if the original chain has length $2^{n-1}+1$, then the new chain has length $2^n + 1$, as desired. This construction yields a chain of length 129 when $S = \{1, 2, \dots, 8\}$.

Remark. Here is the construction for $n = 8$ in its full glory.

345678	1	235678	4	125678	3	145678	2	5678
34	15678	23	45678	12	35678	14	678	
345	1678	235	4678	125	3678	145	2678	5
34678	15	23678	45	12678	35	78		
3456	178	2356	478	1256	378	1456	278	56
3478	156	2378	456	1278	356	1478	6	
34578	16	23578	46	12578	36	14578	26	578
346	1578	236	4578	126	8			
34567	18	23567	48	12567	38	14567	28	567
348	1567	238	4567	128	3567	148	67	
3458	167	2358	467	1258	367	1458	267	58
3467	158	2367	458	1267	358	7		
34568	17	23568	47	12568	37	14568	27	568
347	1568	237	4568	127	3568	147	68	
3457	168	2357	468	1257	368	1457	268	57
3468	157	2368	457	1268				

§1.2 USAMO 2016/2, proposed by Kiran Kedlaya

Available online at https://aops.com/community/p6213627.

Problem statement

Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

We show the exponent of any given prime p is nonnegative in the expression. Recall that the exponent of p in $n!$ is equal to $\sum_{i \ge 1} \lfloor n/p^i \rfloor$. In light of this, it suffices to show that for any prime power q, we have

$$\left\lfloor \frac{k^2}{q} \right\rfloor + \sum_{j=0}^{k-1} \left\lfloor \frac{j}{q} \right\rfloor \ge \sum_{j=0}^{k-1} \left\lfloor \frac{j+k}{q} \right\rfloor$$

Since both sides are integers, we show

$$\left\lfloor \frac{k^2}{q} \right\rfloor + \sum_{j=0}^{k-1} \left\lfloor \frac{j}{q} \right\rfloor > -1 + \sum_{j=0}^{k-1} \left\lfloor \frac{j+k}{q} \right\rfloor.$$

If we denote by $\{x\}$ the fractional part of x, then $\lfloor x \rfloor = x - \{x\}$ so it's equivalent to

$$\left\{ \frac{k^2}{q} \right\} + \sum_{j=0}^{k-1} \left\{ \frac{j}{q} \right\} < 1 + \sum_{j=0}^{k-1} \left\{ \frac{j+k}{q} \right\}.$$

However, the sum of remainders when $0, 1, \dots, k-1$ are taken modulo q is easily seen to be less than the sum of remainders when $k, k+1, \dots, 2k-1$ are taken modulo q. So

$$\sum_{j=0}^{k-1} \left\{ \frac{j}{q} \right\} \le \sum_{j=0}^{k-1} \left\{ \frac{j+k}{q} \right\}$$

follows, and we are done upon noting $\left\{ k^2/q \right\} < 1$.

§1.3 USAMO 2016/3, proposed by Evan Chen, Telv Cohl

Available online at https://aops.com/community/p6213572.

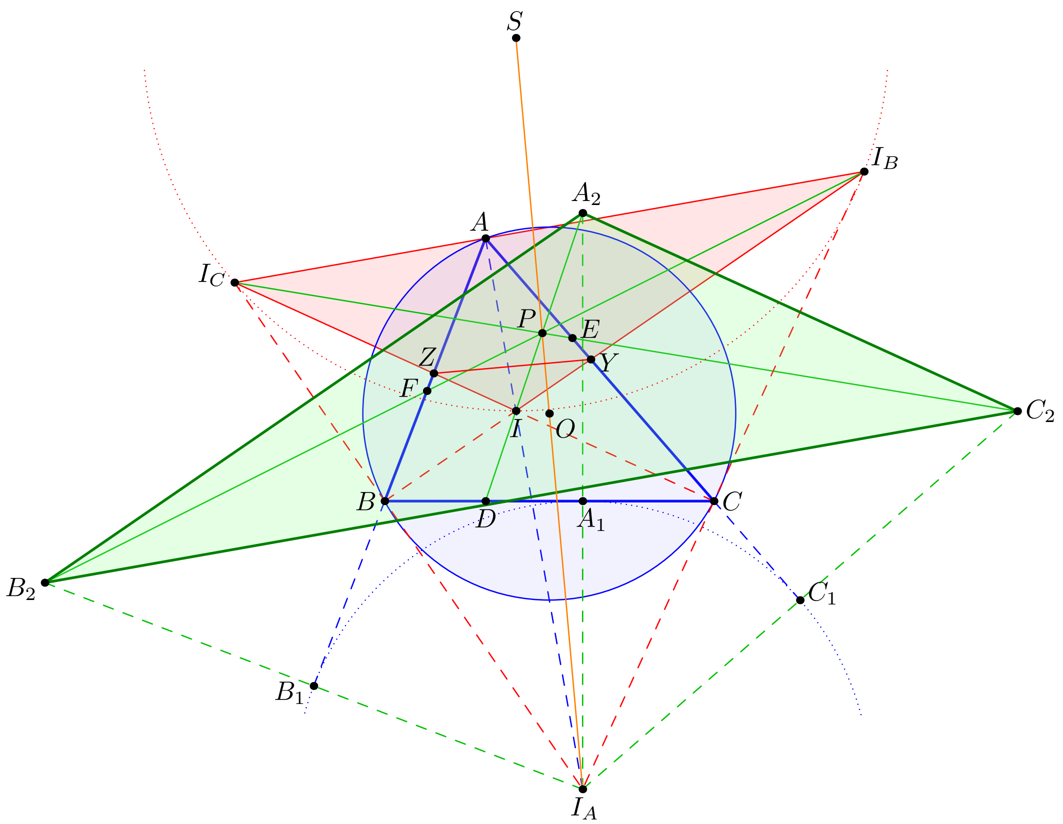
Problem statement

Let ABC be an acute triangle and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines I_BF and I_CE meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

We present two solutions.

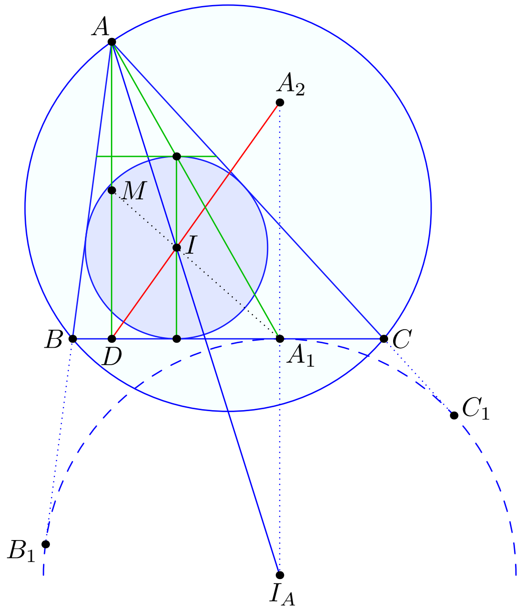
¶ First solution. Let I_A denote the A-excenter and I the incenter. Then let D denote the foot of the altitude from A. Suppose the A-excircle is tangent to $\overline{BC}$, $\overline{AB}$, $\overline{AC}$ at A_1, B_1, C_1 and let A_2, B_2, C_2 denote the reflections of I_A across these points. Let S denote the circumcenter of $\triangle II_BI_C$.



We begin with the following observation:

Claim — Points D, I, A_2 are collinear, as are points E, I_C, C_2 are collinear and points F, I_B, B_2 are collinear.

Proof. This basically follows from the "midpoints of altitudes" lemma. To see D, I, A_2 are collinear, recall first that $\overline{IA_1}$ passes through the midpoint M of $\overline{AD}$.



Now since $\overline{AD} \parallel \overline{I_A A_2}$, and M and A_1 are the midpoints of $\overline{AD}$ and $\overline{I_A A_2}$, it follows from the collinearity of A, I, I_A that D, I, A_2 are collinear as well.

The other two claims follow in a dual fashion. For example, using the homothety taking the A to C-excircle, we find that $\overline{C_1 I_C}$ bisects the altitude $\overline{BE}$, and since I_C, B, I_A are collinear the same argument now gives I_C, E, C_2 are collinear. The fact that I_B, F, B_2 are collinear is symmetric. $\square$

Observe that $\overline{B_2C_2} \parallel \overline{B_1C_1} \parallel \overline{I_BI_C}$. Proceeding similarly on the other sides, we discover $\triangle II_BI_C$ and $\triangle A_2B_2C_2$ are homothetic. Hence P is the center of this homothety (in particular, D, I, P, A_2 are collinear). Moreover, P lies on the line joining I_A to S, which is the Euler line of $\triangle II_BI_C$, so it passes through the nine-point center of $\triangle II_BI_C$, which is O. Consequently, P, O, I_A are collinear as well.

To finish, we need only prove that $\overline{OS} \perp \overline{YZ}$. In fact, we claim that $\overline{YZ}$ is the radical axis of the circumcircles of $\triangle ABC$ and $\triangle II_BI_C$. Actually, Y is the radical center of these two circumcircles and the circle with diameter $\overline{II_B}$ (which passes through A and C). Analogously Z is the radical center of the circumcircles and the circle with diameter $\overline{II_C}$, and the proof is complete.

¶ Second solution (barycentric, outline, Colin Tang). we are going to use barycentric coordinates to show that the line through O perpendicular to $\overline{YZ}$ is concurrent with $\overline{I_BF}$ and $\overline{I_CE}$.

The displacement vector $\overrightarrow{YZ}$ is proportional to $(a(b-c) : -b(a+c) : c(a+b))$, and so by strong perpendicularity criterion and doing a calculation gives the line

$$x(b-c)bc(a+b+c) + y(a+c)ac(a+b-c) + z(a+b)ab(-a+b-c) = 0.$$

On the other hand, line I_CE has equation

$$0 = \det \begin{bmatrix} a & b & -c \\ S_C & 0 & S_A \\ x & y & z \end{bmatrix} = bS_a \cdot x + (-cS_C - aS_A) \cdot y + (-bS_C) \cdot z$$

and similarly for I_BF. Consequently, concurrence of these lines is equivalent to

$$\det \begin{bmatrix} bS_A & -cS_C - aS_A & -bS_C \\ cS_A & -cS_B & -aS_A - bS_B \\ (b-c)bc(a+b+c) & (a+c)ac(a+b-c) & (a+b)ab(-a+b-c) \end{bmatrix} = 0$$

which is a computation.

¶ Authorship comments. I was intrigued by a Taiwan TST problem which implied that, in the configuration above, $\angle I_B D I_C$ was bisected by $\overline{DA}$. This motivated me to draw all three properties above where I_A and P were isogonal conjugates with respect to DEF. After playing around with this picture for a long time, I finally noticed that O was on line PI_A. (So the original was to show that $I_B F$, $I_C E$, DA_2 concurrent). Eventually I finally noticed in the picture that PI_A actually passed through the circumcenter of ABC as well. This took me many hours to prove.

The final restatement (which follows quickly from P, O, I_A collinear) was discovered by Telv Cohl when I showed him the problem.

§2 Solutions to Day 2

§2.1 USAMO 2016/4, proposed by Titu Andreescu

Available online at https://aops.com/community/p6220308.

Problem statement

Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

We claim that the only two functions satisfying the requirements are $f(x) \equiv 0$ and $f(x) \equiv x^2$. These work.

First, taking $x = y = 0$ in the given yields $f(0) = 0$, and then taking $x = 0$ gives $f(y)f(-y) = f(y)^2$. So also $f(-y)^2 = f(y)f(-y)$, from which we conclude f is even. Then taking $x = -y$ gives

$$\forall x \in \mathbb{R} : \qquad f(x) = x^2 \qquad \text{or} \qquad f(4x) = 0 \qquad (\bigstar)$$

for all x.

Remark. Note that an example of a function satisfying $(\bigstar)$ is

$$f(x) = \begin{cases} x^2 & \text{if } |x| < 1 \\ 1 - \cos\left(\frac{\pi}{2} \cdot x^{1337}\right) & \text{if } 1 \le |x| < 4 \\ 0 & \text{if } |x| \ge 4. \end{cases}$$

So, yes, we are currently in a world of trouble, still. (This function is even continuous; I bring this up to emphasize that "continuity" is completely unrelated to the issue at hand.)

Now we claim

Claim — $f(z) = 0 \iff f(2z) = 0 \qquad (\spadesuit)$.

Proof. Let $(x, y) = (3t, t)$ in the given to get

$$\left(f(t) + 3t^2\right) f(8t) = f(4t)^2.$$

Now if $f(4t) \neq 0$ (in particular, $t \neq 0$), then $f(8t) \neq 0$. Thus we have $(\spadesuit)$ in the reverse direction.

Then $f(4t) \neq 0 \overset{(\bigstar)}{\Longrightarrow} f(t) = t^2 \neq 0 \overset{(\spadesuit)}{\Longrightarrow} f(2t) \neq 0$ implies the forwards direction, the last step being the reverse direction $(\spadesuit)$. $\square$

By putting together $(\bigstar)$ and $(\spadesuit)$ we finally get

$$\forall x \in \mathbb{R} : \qquad f(x) = x^2 \qquad \text{or} \qquad f(x) = 0 \qquad (\heartsuit)$$

We are now ready to approach the main problem. Assume there's an $a \neq 0$ for which $f(a) = 0$; we show that $f \equiv 0$.

Let $b \in \mathbb{R}$ be given. Since f is even, we can assume without loss of generality that $a, b > 0$. Also, note that $f(x) \ge 0$ for all x by $(\heartsuit)$. By using $(\spadesuit)$ we can generate $c > b$

such that $f(c) = 0$ by taking $c = 2^n a$ for a large enough integer n. Now, select $x, y > 0$ such that $x - 3y = b$ and $x + y = c$. That is,

$$(x, y) = \left(\frac{3c + b}{4}, \frac{c - b}{4} \right).$$

Substitution into the original equation gives

$$0 = (f(x) + xy)\, f(b) + (f(y) + xy)\, f(3x - y) \geq (f(x) + xy)\, f(b).$$

But since $f(b) \geq 0$, it follows $f(b) = 0$, as desired.

§2.2 USAMO 2016/5, proposed by Ivan Borsenco

Available online at https://aops.com/community/p6220306.

Problem statement

An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overline{MN}$ and $\overline{PQ}$. Denote by ℓ the angle bisector of $\angle MSQ$.

Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

¶ **First solution (complex).** In fact, we only need $AM = AQ = NP$ and $MN = QP$.

We use complex numbers with ABC the unit circle, assuming WLOG that A, B, C are labeled counterclockwise. Let x, y, z be the complex numbers corresponding to the arc midpoints of BC, CA, AB, respectively; thus $x + y + z$ is the incenter of $\triangle ABC$. Finally, let $s > 0$ be the side length of $AM = AQ = NP$.

Then, since $MA = s$ and $MA \perp OZ$, it follows that

$$m - a = i \cdot sz.$$

Similarly, $p - n = i \cdot sy$ and $a - q = i \cdot sx$, so summing these up gives

$$i \cdot s(x + y + z) = (p - q) + (m - n) = (m - n) - (q - p).$$

Since $MN = PQ$, the argument of $(m - n) - (q - p)$ is along the external angle bisector of the angle formed, which is perpendicular to ℓ. On the other hand, $x + y + z$ is oriented in the same direction as OI, as desired.

¶ **Second solution (trig, Danielle Wang).** Let δ and ϵ denote $\angle MNB$ and $\angle CPQ$. Also, assume $AMNPQ$ has side length 1.

In what follows, assume $AB < AC$. First, we note that

$$\begin{aligned} BN &= (c - 1)\cos B + \cos\delta, \\ CP &= (b - 1)\cos C + \cos\epsilon, \text{ and} \\ a &= 1 + BN + CP \end{aligned}$$

from which it follows that

$$\cos\delta + \cos\epsilon = \cos B + \cos C - 1$$

Also, by the Law of Sines, we have $\frac{c-1}{\sin\delta} = \frac{1}{\sin B}$ and similarly on triangle CPQ, and from this we deduce

$$\sin\epsilon - \sin\delta = \sin B - \sin C.$$

The sum-to-product formulas

$$\begin{aligned} \sin\epsilon - \sin\delta &= 2\cos\left(\frac{\epsilon+\delta}{2}\right)\sin\left(\frac{\epsilon-\delta}{2}\right) \\ \cos\epsilon - \cos\delta &= 2\cos\left(\frac{\epsilon+\delta}{2}\right)\cos\left(\frac{\epsilon-\delta}{2}\right) \end{aligned}$$

give us

$$\tan\left(\frac{\epsilon-\delta}{2}\right)=\frac{\sin\epsilon-\sin\delta}{\cos\epsilon-\cos\delta}=\frac{\sin B-\sin C}{\cos B+\cos C-1}.$$

Now note that ℓ makes an angle of $\frac{1}{2}(\pi+\epsilon-\delta)$ with line BC. Moreover, if line OI intersects line BC with angle φ then

$$\tan\varphi=\frac{r-R\cos A}{\frac{1}{2}(b-c)}.$$

So in order to prove the result, we only need to check that

$$\frac{r-R\cos A}{\frac{1}{2}(b-c)}=\frac{\cos B+\cos C+1}{\sin B-\sin C}.$$

Using the fact that $b=2R\sin B$, $c=2R\sin C$, this reduces to the fact that $r/R+1=\cos A+\cos B+\cos C$, which is the so-called Carnot theorem.

§2.3 USAMO 2016/6, proposed by Gabriel Carroll

Available online at https://aops.com/community/p6220302.

Problem statement

Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard. The wizard has $2n$ cards; for each $i = 1, \dots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order. You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is *winnable* if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds. For which values of n and k is the game winnable?

The game is winnable if and only if $k < n$.

First, suppose $2 \leq k < n$. Query the cards in positions $\{1, \dots, k\}$, then $\{2, \dots, k+1\}$, and so on, up to $\{2n-k+1, 2n\}$. Indeed, by taking the difference of the ith and $(i+1)$st query, we can deduce the value of the ith card, for $1 \leq i \leq 2n-k$. (This is possible because the cards are flipped face up before they are re-shuffled, so even if two adjacent queries return the same set, one can still determine the ith card. It is possible to solve the problem even without the flipped information, though.) If $k \leq n$, this is more than n cards, so we can find a matching pair.

For $k = n$ we remark the following: at each turn after the first, assuming one has not won, there are n cards representing each of the n values exactly once, such that the player has no information about the order of those n cards. We claim that consequently the player cannot guarantee victory. Indeed, let S denote this set of n cards, and $\overline{S}$ the other n cards. The player will never win by picking only cards in S or $\overline{S}$. Also, if the player selects some cards in S and some cards in $\overline{S}$, then it is possible that the choice of cards in S is exactly the complement of those selected from $\overline{S}$; the strategy cannot prevent this since the player has no information on S. This implies the result.