USAMO 2013 Solution Notes

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This is a compilation of solutions for the 2013 USAMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

2. For a positive integer $n \ge 3$ plot n equally spaced points around a circle. Label one of them A, and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \ge 4$.

3. Let n be a positive integer. There are $\frac{n(n+1)}{2}$ tokens, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n tokens. Initially, each token has the white side up. An operation is to choose a line parallel to the sides of the triangle, and flip all the token on that line. A configuration is called admissible if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

4. Find all real numbers $x, y, z \ge 1$ satisfying
$$\min\left(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\right) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

5. Let m and n be positive integers. Prove that there exists a positive integer c such that cm and cn have the same nonzero decimal digits.

6. Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then
$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

§1 Solutions to Day 1

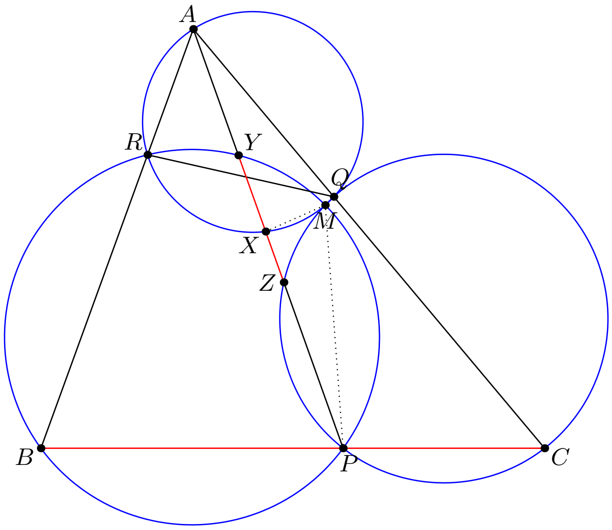
§1.1 USAMO 2013/1, proposed by Zuming Feng

Available online at https://aops.com/community/p3041822.

Problem statement

In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

Let M be the concurrence point of ω_A, ω_B, ω_C (by Miquel's theorem).



Then M is the center of a spiral similarity sending $\overline{YZ}$ to $\overline{BC}$. So it suffices to show that this spiral similarity also sends X to P, but

$$\measuredangle MXY = \measuredangle MXA = \measuredangle MRA = \measuredangle MRB = \measuredangle MPB$$

so this follows.

§1.2 USAMO 2013/2, proposed by Kiran Kedlaya

Available online at https://aops.com/community/p3041823.

Problem statement

For a positive integer $n \ge 3$ plot n equally spaced points around a circle. Label one of them A, and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \ge 4$.

We present two similar approaches.

¶ First solution. Imagine the counter is moving along the set $S = \{0, 1, \dots, 2n\}$ instead, starting at 0 and ending at $2n$, in jumps of length 1 and 2. We can then record the sequence of moves as a matrix of the form

$$\begin{bmatrix} p_0 & p_1 & p_2 & \dots & p_{n-1} & p_n \\ p_n & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & p_{2n} \end{bmatrix}$$

where $p_i = 1$ if the point i was visited by the counter, and $p_i = 0$ if the point was not visited by the counter. Note that $p_0 = p_{2n} = 1$ and the upper-right and lower-left entries are equal. Then, the problem amounts to finding the number of such matrices which avoid the contiguous submatrices

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which correspond to forbidding jumps of length greater than 2, repeating a length 2 jump and repeating a length 1 jump.

We give a nice symmetric phrasing suggested by `fclvbfm934` at https://aops.com/community/p27834267. If we focus on just the three possible column vectors that appear, say

$$\mathbf{u} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{w} := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

then we can instead describe valid matrices as sequences of $n+1$ such column vectors, where no two column vectors are adjacent, and where the boundary condition is that

- either we start with $\mathbf{u}$ and end with $\mathbf{v}$, or
- either we start with $\mathbf{w}$ and end with $\mathbf{w}$.

Let x_n and y_n denote the number of such $2 \times (n+1)$ matrices. (Hence $a_n = x_n + y_n$.) But owing to the symmetry of the setup with $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, we could instead view x_n and y_n as the number of $2 \times (n+1)$ matrices for a fixed starting first column whose final column is the same/different. So we have the recursions

$$\begin{aligned} x_{n+1} &= x_n + y_n \\ y_{n+1} &= 2x_n. \end{aligned}$$

We also have that

$$2x_n + y_n = 2^n$$

which may either be proved directly from the recursions (using $x_1 = 1$ and $y_1 = 0$), or by noting the left-hand side counts the total number of sequences of $n+1$ column vectors with no restrictions on the final column at all (in which case there are simply 2 choices for each of the n columns after the first one). Thus,

$$\begin{aligned} a_{n+1} + a_n &= (x_{n+1} + y_{n+1}) + (x_n + y_n) \\ &= ((x_n + y_n) + 2x_n) + (x_n + y_n) \\ &= 2(2x_n + y_n) = 2^{n+1} \end{aligned}$$

as needed.

¶ Second (longer) solution. If one does not notice the nice rephrasing with **u**, **v**, **w** above, one may still proceed with the following direct calculation. Retain the notation of

$$\begin{bmatrix} p_0 & p_1 & p_2 & \dots & p_{n-1} & p_n \\ p_n & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & p_{2n} \end{bmatrix}$$

described earlier. We will for now ignore the boundary conditions. Instead we say a $2 \times m$ matrix is *silver* ($m \ge 2$) if it avoids the three shapes above. We consider three types of silver matrices (essentially doing casework on the last column):

- *type B matrices*, of the shape $\begin{bmatrix} 1 & \dots & 1 \\ 0 & \dots & 0 \end{bmatrix}$
- *type C matrices*, of the shape $\begin{bmatrix} 1 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}$.
- *type D matrices*, of the shape $\begin{bmatrix} 1 & \dots & 1 \\ 0 & \dots & 1 \end{bmatrix}$.

We let b_m, c_m, d_m denote matrices of each type, of size $2 \times m$, and claim the following two recursions for $m \ge 4$:

$$\begin{aligned} b_m &= c_{m-1} + d_{m-1} \\ c_m &= b_{m-1} + d_{m-1} \\ d_m &= b_{m-1} + c_{m-1}. \end{aligned}$$

Indeed, if we delete the last column of a type B matrix and consider what used to be the second-to-last column, we find that it is either type C or type D. This establishes the first recursion and the others are analogous.

Note that $b_2 = 0$ and $c_2 = d_2 = 1$. So using this recursion, the first few values are

m	2	3	4	5	6	7	8
b_m	0	2	2	6	10	22	42
c_m	1	1	3	5	11	21	43
d_m	1	1	3	5	11	21	43

and a calculation gives $b_m = \frac{2^{m-1}+2(-1)^{m-1}}{3}$, $c_m = d_m = \frac{2^{m-1}-(-1)^{m-1}}{3}$.

We now relate a_n to b_m, c_m, d_m. Observe that a matrix as described in the problem is a silver matrix of one of two forms:

$$\begin{bmatrix} 1 & p_1 & p_2 & \dots & p_{n-1} & 0 \\ 0 & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & 1 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1 & p_1 & p_2 & \dots & p_{n-1} & 1 \\ 1 & p_{n+1} & p_{n+2} & \dots & p_{2n-1} & 1 \end{bmatrix}.$$

There are c_{n+1} matrices of the first form. Moreover, there are $2d_n$ matrices of the second form (to see this, delete the first column; we either get a type-D matrix or an upside-down type-D matrix). Thus we get

$$a_n = c_{n+1} + 2d_n = \frac{2^{n+1} + (-1)^{n+1}}{3}.$$

This implies the result.

Remark. The two solutions are closely related. In fact, $c_n = x_{n-1}$ and $b_n = y_{n-1}$. So the second solution is really the same as the first solution, except the symmetry of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ was not noticed, thus requiring a third recursion to handle all the cases manually.

§1.3 USAMO 2013/3, proposed by Warut Suksompong

Available online at https://aops.com/community/p3041827.

Problem statement

Let n be a positive integer. There are $\frac{n(n+1)}{2}$ tokens, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n tokens. Initially, each token has the white side up. An operation is to choose a line parallel to the sides of the triangle, and flip all the token on that line. A configuration is called admissible if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

The answer is

$$\max_C f(C) = \begin{cases} 6k & n = 4k \\ 6k+1 & n = 4k+1 \\ 6k+2 & n = 4k+2 \\ 6k+3 & n = 4k+3. \end{cases}$$

The main point of the problem is actually to determine all linear dependencies among the $3n$ possible moves (since the moves commute and applying a move twice is the same as doing nothing). In what follows, assume $n > 1$ for convenience.

To this end, we consider sequences of operations as additive vectors in $v \in \mathbb{F}_2^{3n}$, with the linear map $T \colon \mathbb{F}_2^{3n} \to \mathbb{F}_2^{\frac{1}{2}n(n+1)}$ denoting the result of applying a vector v. We in particular focus on the following four vectors.

- Three vectors x, y, z are defined by choosing all n lines parallel to one axis. Note $T(x) = T(y) = T(z) = \mathbf{1}$ (i.e. these vectors flip all tokens).
- The vector θ which toggles all lines with an even number of tokens. One can check that $T(\theta) = \mathbf{0}$. (Easiest to guess from $n = 2$ and $n = 3$ case.) One amusing proof that this works is to use Vivani's theorem: in an equilateral triangle ABC, the sum of distances from an interior point P to the three sides is equal.

The main claim is:

Claim — For $n \ge 2$, the kernel of T has exactly eight elements, namely $\{\mathbf{0}, x+y, y+z, z+x, \theta, \theta+x+y, \theta+y+z, \theta+z+x\}$.

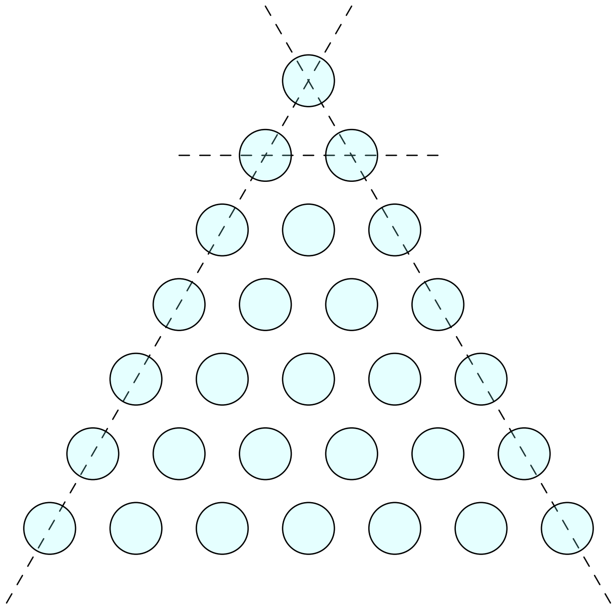
Proof. Suppose $T(v) = 0$.

- If v uses the y-move of length n, then we replace v with $v + (x+y)$ to obtain a vector in the kernel not using the y-move of length n.
- If v uses the z-move of length n, then we replace v with $v + (x+z)$ to obtain a vector in the kernel not using the z-move of length n.
- If v uses the x-move of length 2, then

  – if n is odd, replace v with $v+\theta$;
  – if n is even, replace v with $v+(\theta+y+z)$

to obtain a vector in the kernel not using the x-move of length 2.

A picture is shown below, with the unused rows being dotted.



Then, it is easy to check inductively that v must now be the zero vector, after the replacements. The idea is that for each token t, if two of the moves involving t are unused, so is the third, and in this way we can show all rows are unused. Thus the original v was in the kernel we described.

(An alternative proof by induction is feasible too; as a sequence of movings which does not affect the top n rows also does not affect the to $n-1$ rows.) $\square$

Then problem is a coordinate bash, since given any v we now know exactly which vectors w have $T(v)=T(w)$, so given any admissible configuration C one can exactly compute $f(C)$ as the minimum of eight values.

To be explicit, we could represent a vector v as

$$v \longleftrightarrow (a_1, a_2, b_1, b_2, c_1, c_2)$$

where a_1 is the number of 1's in odd x-indices, a_2 number of 1's in even x-indices. Then for example

$$\begin{aligned} v &\longleftrightarrow (a_1, a_2, b_1, b_2, c_1, c_2) \\ v+x+y &\longleftrightarrow \left(\left\lceil \frac{n}{2} \right\rceil - a_1, \left\lfloor \frac{n}{2} \right\rfloor - a_2, \left\lceil \frac{n}{2} \right\rceil - b_1, \left\lfloor \frac{n}{2} \right\rfloor - b_2, c_1, c_2\right) \\ v+\theta &\longleftrightarrow \left(a_1, \left\lfloor \frac{n}{2} \right\rfloor - a_2, b_1, \left\lfloor \frac{n}{2} \right\rfloor - b_2, c_1, \left\lfloor \frac{n}{2} \right\rfloor - c_2\right) \qquad \vdots \end{aligned}$$

and $f(T(v))$ is the smallest sum of the six numbers across all eight 6-tuples. So you expect to answer about $\frac{3}{2}n$ if all things are about $n/4$. The details are too annoying to reproduce here, so they are omitted.

§2 Solutions to Day 2

§2.1 USAMO 2013/4, proposed by Titu Andreescu

Available online at https://aops.com/community/p3043752.

Problem statement

Find all real numbers $x, y, z \geq 1$ satisfying

$$\min\left(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\right) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Set $x = 1+a$, $y = 1+b$, $z = 1+c$ which eliminates the $x, y, z \geq 1$ condition. Assume without loss of generality that $a \leq b \leq c$. Then the given equation rewrites as

$$\sqrt{(1+a)\left(1+(1+b)(1+c)\right)} = \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

In fact, we are going to prove the left-hand side always exceeds the right-hand side, and then determine the equality cases. We have:

$$\begin{aligned}(1+a)\left(1+(1+b)(1+c)\right) &= (a+1)\left(1+(b+1)(1+c)\right) \\ &\leq (a+1)\left(1+\left(\sqrt{b}+\sqrt{c}\right)^2\right) \\ &\leq \left(\sqrt{a}+\left(\sqrt{b}+\sqrt{c}\right)\right)^2\end{aligned}$$

by two applications of Cauchy-Schwarz.

Equality holds if $bc = 1$ and $1/a = \sqrt{b}+\sqrt{c}$. Letting $c = t^2$ for $t \geq 1$, we recover $b = t^{-2} \leq t^2$ and $a = \frac{1}{t+1/t} \leq t^2$.

Hence the solution set is

$$(x, y, z) = \left(1 + \left(\frac{t}{t^2+1}\right)^2, 1 + \frac{1}{t^2}, 1 + t^2\right)$$

and permutations, for any $t > 0$.

§2.2 USAMO 2013/5, proposed by Richard Stong

Available online at https://aops.com/community/p3043754.

Problem statement

Let m and n be positive integers. Prove that there exists a positive integer c such that cm and cn have the same nonzero decimal digits.

One-line spoiler: 142857.

More verbosely, the idea is to look at the decimal representation of $1/D$, m/D, n/D for a suitable denominator D, which have a "cyclic shift" property in which the digits of n/D are the digits of m/D shifted by 3.

Remark (An example to follow along)**.** Here is an example to follow along in the subsequent proof If $m = 4$ and $n = 23$ then the magic numbers $e = 3$ and $D = 41$ obey

$$10^3 \cdot \frac{4}{41} = 97 + \frac{23}{41}.$$

The idea is that

$$\begin{aligned} \frac{1}{41} &= 0.\overline{02439} \\ \frac{4}{41} &= 0.\overline{09756} \\ \frac{23}{41} &= 0.\overline{56097} \end{aligned}$$

and so $c = 2349$ works; we get $4c = 9756$ and $23c = 56097$ which are cyclic shifts of each other by 3 places (with some leading zeros appended).

Here is the one to use:

Claim — There exists positive integers D and e such that $\gcd(D, 10) = 1$, $D > \max(m, n)$, and moreover

$$\frac{10^e m - n}{D} \in \mathbb{Z}.$$

Proof. Suppose we pick some exponent e and define the number

$$A = 10^e n - m.$$

Let $r = \nu_2(m)$ and $s = \nu_5(m)$. As long as $e > \max(r, s)$ we have $\nu_2(A) = r$ and $\nu_5(A) = s$, too. Now choose any $e > \max(r, s)$ big enough that $A > 2^r 5^s \max(m, n)$ also holds. Then the number $D = \frac{A}{2^r 5^s}$ works; the first two properties hold by construction and

$$10^e \cdot \frac{n}{D} - \frac{m}{D} = \frac{A}{D} = 2^r 5^s$$

is an integer. □

Remark (For people who like obscure theorems)**.** Kobayashi's theorem implies we can actually pick D to be prime.

Now we take c to be the number under the bar of $1/D$ (leading zeros removed). Then the decimal representation of $\frac{m}{D}$ is the decimal representation of cm repeated (possibly including leading zeros). Similarly, $\frac{n}{D}$ has the decimal representation of cm repeated (possibly including leading zeros). Finally, since

$$10^e \cdot \frac{m}{D} - \frac{n}{D} \text{ is an integer}$$

it follows that these repeating decimal representations are rotations of each other by e places, so in particular they have the same number of nonzero digits.

Remark. Many students tried to find a D satisfying the stronger hypothesis that $1/D$, $2/D$, ..., $(D-1)/D$ are cyclic shifts of each other. For example, this holds in the famous $D = 7$ case.

The official USAMO 2013 solutions try to do this by proving that 10 is a primitive root modulo 7^e for each $e \ge 1$, by Hensel lifting lemma. I think this argument is actually *incorrect*, because it breaks if either m or n are divisible by 7. Put bluntly, $\frac{7}{49}$ and $\frac{8}{49}$ are not shifts of each other.

One may be tempted to resort to using large primes D rather than powers of 7 to deal with this issue. However it is an open conjecture (a special case of Artin's primitive root conjecture) whether or not $10 \pmod p$ is primitive infinitely often, which is the necessary conjecture so this is harder than it seems.

§2.3 USAMO 2013/6, proposed by Titu Andreescu, Cosmin Pohoata

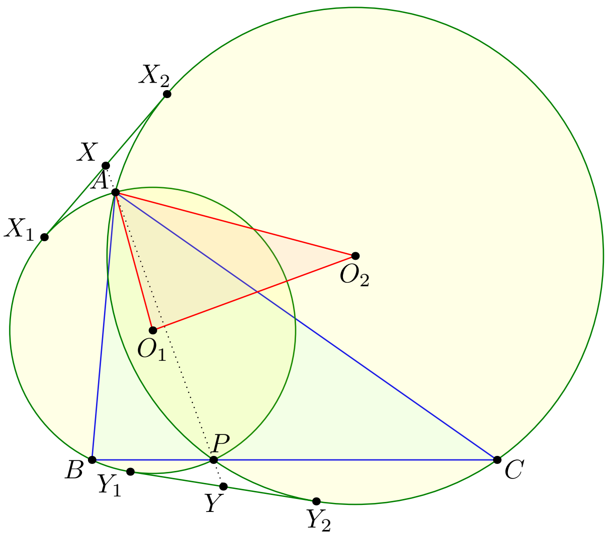
Available online at https://aops.com/community/p3043749.

Problem statement

Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

Let O_1 and O_2 denote the circumcenters of PAB and PAC. The main idea is to notice that $\triangle ABC$ and $\triangle AO_1O_2$ are spirally similar.



Claim (Salmon theorem) — We have $\triangle ABC \overset{+}{\sim} \triangle AO_1O_2$.

Proof. We first claim $\triangle AO_1B \overset{+}{\sim} \triangle AO_2C$. Assume without loss of generality that $\angle APB \le 90^\circ$. Then

$$\angle AO_1B = 2\angle APB$$

but

$$\angle AO_2C = 2\left(180 - \angle APC\right) = 2\angle ABP.$$

Hence $\angle AO_1B = \angle AO_2C$. Moreover, both triangles are isosceles, establishing the first similarity. The second part follows from spiral similarities coming in pairs. $\square$

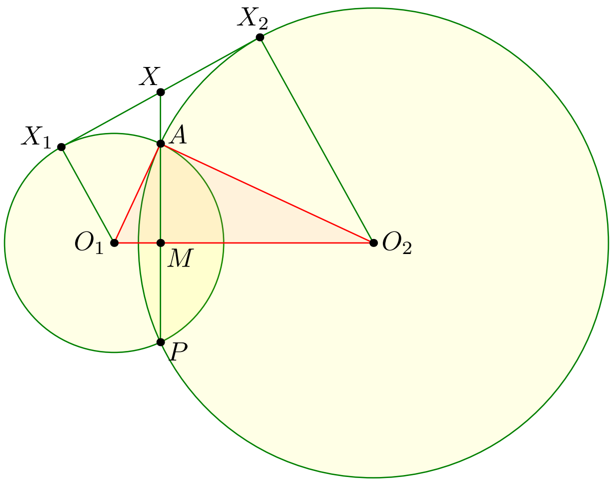
Claim — We always have

$$\left(\frac{PA}{XY}\right)^2 = 1 - \left(\frac{a}{b+c}\right)^2.$$

(In particular, this does not depend on P.)

Proof. We now delete the points B and C and remember only the fact that $\triangle AO_1O_2$ has angles A, B, C. The rest is a computation and several approaches are possible.

Without loss of generality A is closer to X than Y, and let the common tangents be $\overline{X_1X_2}$ and $\overline{Y_1Y_2}$. We'll perform the main calculation with the convenient scaling $O_BO_C = a$, $AO_C = b$, and $AO_B = c$. Let B_1 and C_1 be the tangency points of X, and let $h = AM$ be the height of $\triangle AO_BO_C$.



Note that by Power of a Point, we have $XX_1^2 = XX_2^2 = XM^2 - h^2$. Also, by Pythagorean theorem we easily obtain $X_1X_2 = a^2 - (b-c)^2$. So putting these together gives

$$XM^2 - h^2 = \frac{a^2-(b-c)^2}{4} = \frac{(a+b-c)(a-b+c)}{4} = (s-b)(s-c).$$

Therefore, we have

Then

$$\begin{aligned}\frac{XM^2}{h^2} &= 1 + \frac{(s-b)(s-c)}{h^2} = 1 + \frac{a^2(s-b)(s-c)}{a^2h^2} \\ &= 1 + \frac{a^2(s-b)(s-c)}{4s(s-a)(s-b)(s-c)} = 1 + \frac{a^2}{4s(s-a)} \\ &= 1 + \frac{a^2}{(b+c)^2-a^2} = \frac{(b+c)^2}{(b+c)^2-a^2}.\end{aligned}$$

Thus

$$\left(\frac{PA}{XY}\right)^2 = \left(\frac{h}{XM}\right)^2 = 1 - \left(\frac{a}{b+c}\right)^2. \qquad \square$$

To finish, note that when P is the foot of the $\angle A$-bisector, we necessarily have

$$\frac{PB\cdot PC}{AB\cdot AC} = \frac{\left(\frac{b}{b+c}a\right)\left(\frac{c}{b+c}a\right)}{bc} = \left(\frac{a}{b+c}\right)^2.$$

Since there are clearly at most two solutions as $\frac{PA}{XY}$ is fixed, these are the only two solutions.