USAMO 2010 Solution Notes

Evan Chen《陳誼廷》

15 April 2024

This is a compilation of solutions for the 2010 USAMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

0 Problems **2**

1 Solutions to Day 1 **3**

1.1 USAMO 2010/1, proposed by Titu Andreescu 3

1.2 USAMO 2010/2, proposed by David Speyer 5

1.3 USAMO 2010/3, proposed by Gabriel Carroll 6

2 Solutions to Day 2 **7**

2.1 USAMO 2010/4, proposed by Zuming Feng 7

2.2 USAMO 2010/5, proposed by Titu Andreescu 8

2.3 USAMO 2010/6, proposed by Gerhard Woeginger 9

§0 Problems

1. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

2. There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

3. The 2010 positive real numbers $a_1, a_2, \dots, a_{2010}$ satisfy the inequality $a_i a_j \le i + j$ for all $1 \le i < j \le 2010$. Determine, with proof, the largest possible value of the product $a_1 a_2 \dots a_{2010}$.

4. Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

5. Let $q = \frac{3p-5}{2}$ where p is an odd prime, and let

$$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)}.$$

   Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers m and n, then $m - n$ is divisible by p.

6. There are 68 ordered pairs (not necessarily distinct) of nonzero integers on a blackboard. It's known that for no integer k does both (k, k) and $(-k, -k)$ appear. A student erases some of the 136 integers such that no two erased integers have sum zero, and scores one point for each ordered pair with at least one erased integer. What is the maximum possible score the student can guarantee?

§1 Solutions to Day 1

§1.1 USAMO 2010/1, proposed by Titu Andreescu

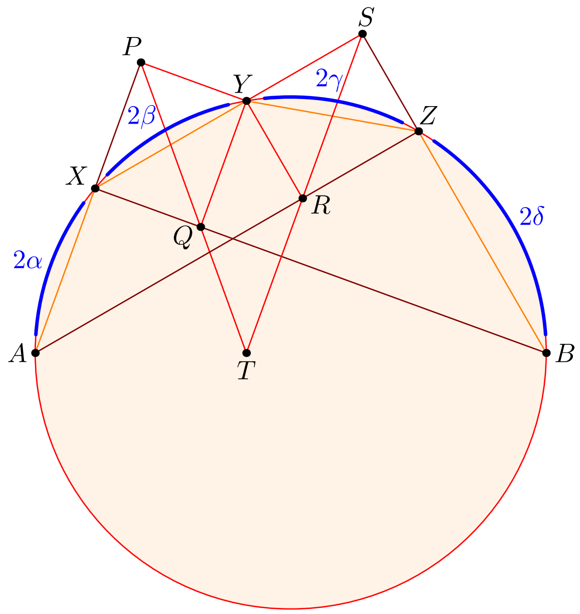
Available online at https://aops.com/community/p1860802.

Problem statement

Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

We present two possible approaches. The first approach is just "bare-hands" angle chasing. The second approach requires more insight but makes it clearer what is going on; it shows the intersection point of lines PQ and RS is the foot from the altitude from Y to AB using Simson lines. The second approach also has the advantage that it works even if $\overline{AB}$ is not a diameter of the circle.

¶ First approach using angle chasing. Define $T = \overline{PQ} \cap \overline{RS}$. Also, let 2α, 2β, 2γ, 2δ denote the measures of arcs $\widehat{AX}$, $\widehat{XY}$, $\widehat{YZ}$, $\widehat{ZB}$, respectively, so that $\alpha + \beta + \gamma + \delta = 90^\circ$.



We now compute the following angles:

$$\begin{aligned}
\angle SRY &= \angle SZY = 90^\circ - \angle YZA = 90^\circ - (\alpha + \beta) \\
\angle YQP &= \angle YXP = 90^\circ - \angle BXY = 90^\circ - (\gamma + \delta) \\
\angle QYR &= 180^\circ - \angle(\overline{ZR}, \overline{QX}) = 180^\circ - \frac{2\beta + 2\gamma + 180^\circ}{2} = 90^\circ - (\beta + \gamma).
\end{aligned}$$

Hence, we can then compute

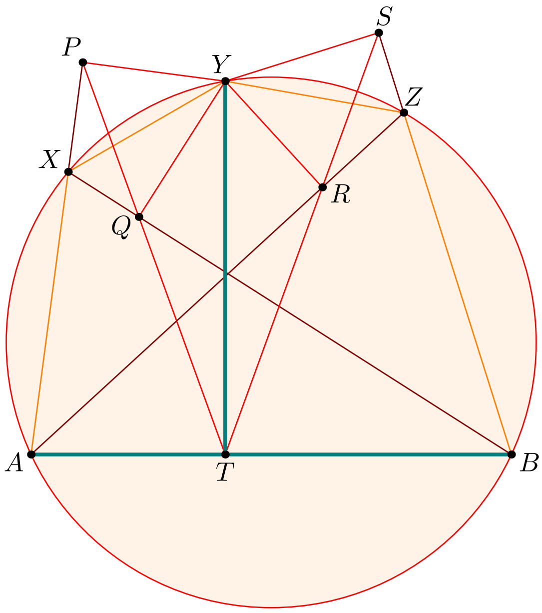
$$\begin{aligned}
\angle RTQ &= 360^\circ - (\angle QYR + (180^\circ - \angle SRY) + (180^\circ - \angle YQP)) \\
&= \angle SRY + \angle YQP - \angle QYR
\end{aligned}$$

$$\begin{aligned} &= (90^\circ - (\alpha+\beta)) + (90^\circ - (\gamma+\delta)) - (90^\circ - (\beta+\gamma)) \\ &= 90^\circ - (\alpha+\delta) \\ &= \beta+\gamma. \end{aligned}$$

Since $\angle XOZ = \frac{2\beta+2\gamma}{2} = \beta+\gamma$, the proof is complete.

¶ Second approach using Simson lines, ignoring the diameter condition. In this solution, we will ignore the condition that $\overline{AB}$ is a diameter; the solution works equally well without it, as long as O is redefined as the center of $(AXYZB)$ instead. We will again show the angle formed by lines PQ and RS is half the measure of $\widehat{XZ}$.

Unlike the previous solution, we instead define T to be the foot from Y to $\overline{AB}$. Then the Simson line of Y with respect to $\triangle XAB$ passes through P, Q, T. Similarly, the Simson line of Y with respect to $\triangle ZAB$ passes through R, S, T. Therefore, point T coincides with $\overline{PQ} \cap \overline{RS}$.



Now it's straightforward to see $APYRT$ is cyclic (in the circle with diameter $\overline{AY}$), and therefore

$$\angle RTY = \angle RAY = \angle ZAY.$$

Similarly,

$$\angle YTQ = \angle YBQ = \angle YBX.$$

Summing these gives $\angle RTQ$ is equal to half the measure of arc $\widehat{XZ}$ as needed.

§1.2 USAMO 2010/2, proposed by David Speyer

Available online at https://aops.com/community/p1860777.

Problem statement

There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \dots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

The main claim is the following observation, which is most motivated in the situation $j - i = 2$.

Claim — The students with heights h_i and h_j switch at most $|j - i| - 1$ times.

Proof. By induction on $d = |j - i|$, assuming $j > i$. For $d = 1$ there is nothing to prove.

For $d \geq 2$, look at only students h_j, h_{i+1} and h_i ignoring all other students. After h_j and h_i switch the first time, the relative ordering of the students must be $h_i \to h_j \to h_{i+1}$. Thereafter h_j must always switch with h_{i+1} before switching with h_i, so the inductive hypothesis applies to give the bound $1 + j - (i + 1) - 1 = j - i - 1$. □

Hence, the number of switches is at most

$$\sum_{1 \leq i < j \leq n} (|j - i| - 1) = \binom{n}{3}.$$

§1.3 USAMO 2010/3, proposed by Gabriel Carroll

Available online at https://aops.com/community/p1860806.

Problem statement

The 2010 positive real numbers $a_1, a_2, \dots, a_{2010}$ satisfy the inequality $a_i a_j \le i + j$ for all $1 \le i < j \le 2010$. Determine, with proof, the largest possible value of the product $a_1 a_2 \dots a_{2010}$.

The answer is $3 \times 7 \times 11 \times \dots \times 4019$, which is clearly an upper bound (and it's not too hard to show this is the lowest number we may obtain by multiplying 1005 equalities together; this is essentially the rearrangement inequality). The tricky part is the construction. Intuitively we want $a_i \approx \sqrt{2i}$, but the details require significant care.

Note that if this is achievable, we will require $a_n a_{n+1} = 2n + 1$ for all odd n. Here are two constructions:

- One can take the sequence such that $a_{2008} a_{2010} = 4018$ and $a_n a_{n+1} = 2n + 1$ for all $n = 1, 2, \dots, 2009$. This can be shown to work by some calculation. As an illustrative example,
$$a_1 a_4 = \frac{a_1 a_2 \cdot a_3 a_4}{a_2 a_3} = \frac{3 \cdot 7}{5} < 5.$$
- In fact one can also take $a_n = \sqrt{2n}$ for all even n (and hence $a_{n-1} = \sqrt{2n} - \frac{1}{\sqrt{2n}}$ for such even n).

Remark. This is a chief example of an "abstract" restriction-based approach. One can motivate it in three steps:

- The bound $3 \cdot 7 \cdot \dots \cdot 4019$ is provably best possible upper bound by pairing the inequalities; also the situation with 2010 replaced by 4 is constructible with bound 21.
- We have $a_n \approx \sqrt{2n}$ heuristically; in fact $a_n = \sqrt{2n}$ satisfies inequalities by AM-GM.
- So we are most worried about $a_i a_j \le i + j$ when $|i - j|$ is small, like $|i - j| = 1$.

I then proceeded to spend five hours on various constructions, but it turns out that the right thing to do was just require $a_k a_{k+1} = 2k + 1$, to make sure these pass: and the problem almost solves itself.

Remark. When 2010 is replaced by 4 it is not too hard to manually write an explicit example: say $a_1 = \frac{\sqrt{3}}{1.1}$, $a_2 = 1.1\sqrt{3}$, $a_3 = \frac{\sqrt{7}}{1.1}$ and $a_4 = 1.1\sqrt{7}$. So this is a reason one might guess that $3 \times 7 \times \dots \times 4019$ can actually be achieved in the large case.

Remark. Victor Wang says: I believe we can actually prove that WLOG (!) assume $a_i a_{i+1} = 2i + 1$ for all i (but there are other ways to motivate that as well, like linear programming after taking logs), which makes things a bit simpler to think about.

§2 Solutions to Day 2

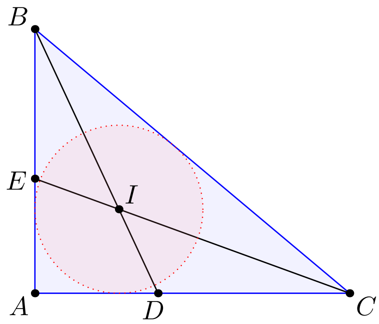
§2.1 USAMO 2010/4, proposed by Zuming Feng

Available online at https://aops.com/community/p1860753.

Problem statement

Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

The answer is no. We prove that it is not even possible that AB, AC, CI, IB are all integers.



First, we claim that $\angle BIC = 135°$. To see why, note that

$$\angle IBC + \angle ICB = \frac{\angle B}{2} + \frac{\angle C}{2} = \frac{90°}{2} = 45°.$$

So, $\angle BIC = 180° - (\angle IBC + \angle ICB) = 135°$, as desired.

We now proceed by contradiction. The Pythagorean theorem implies

$$BC^2 = AB^2 + AC^2$$

and so BC^2 is an integer. However, the law of cosines gives

$$\begin{aligned} BC^2 &= BI^2 + CI^2 - 2BI \cdot CI \cos \angle BIC \\ &= BI^2 + CI^2 + BI \cdot CI \cdot \sqrt{2}. \end{aligned}$$

which is irrational, and this produces the desired contradiction.

§2.2 USAMO 2010/5, proposed by Titu Andreescu

Available online at https://aops.com/community/p1860791.

Problem statement

Let $q = \frac{3p-5}{2}$ where p is an odd prime, and let

$$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)}.$$

Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers m and n, then $m - n$ is divisible by p.

By partial fractions, we have

$$\frac{2}{(3k-1)(3k)(3k+1)} = \frac{1}{3k-1} - \frac{2}{3k} + \frac{1}{3k+1}.$$

Thus

$$\begin{aligned} 2S_q &= \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) + \cdots + \left(\frac{1}{q} - \frac{2}{q+1} + \frac{1}{q+2}\right) \\ &= \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{q+2}\right) - 3\left(\frac{1}{3} + \frac{1}{6} + \cdots + \frac{1}{q+1}\right) \\ &= \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{q+2}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{\frac{q+1}{3}}\right) \\ \implies 2S_q - \frac{1}{p} + 1 &= \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1}\right) + \left(\frac{1}{p+1} + \frac{1}{p+2} \cdots + \frac{1}{q+2}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{\frac{q+1}{3}}\right) \end{aligned}$$

Now we are ready to take modulo p. The given says that $q - p + 2 = \frac{q+1}{3}$, so

$$\begin{aligned} 2S_q - \frac{1}{p} + 1 &= \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1}\right) + \left(\frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{q+2}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{\frac{q+1}{3}}\right) \\ &\equiv \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1}\right) + \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{q-p+2}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{\frac{q+1}{3}}\right) \\ &= \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1} \\ &\equiv 0 \pmod{p}. \end{aligned}$$

So $\frac{1}{p} - 2S_q \equiv 1 \pmod{p}$ which is the desired.

§2.3 USAMO 2010/6, proposed by Gerhard Woeginger

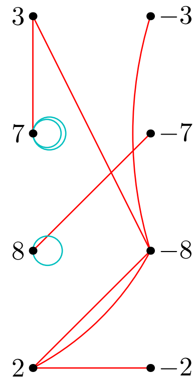
Available online at https://aops.com/community/p1860794.

Problem statement

There are 68 ordered pairs (not necessarily distinct) of nonzero integers on a blackboard. It's known that for no integer k does both (k, k) and $(-k, -k)$ appear. A student erases some of the 136 integers such that no two erased integers have sum zero, and scores one point for each ordered pair with at least one erased integer. What is the maximum possible score the student can guarantee?

The answer is 43.

The structure of this problem is better understood as follows: we construct a multigraph whose vertices are the entries, and the edges are the 68 ordered pairs on the blackboard. To be precise, construct a multigraph G with vertices $a_1, b_1, \dots, a_n, b_n$, with $a_i = -b_i$ for each i. The ordered pairs then correspond to 68 edges in G, with self-loops allowed (WLOG) only for vertices a_i. The student may then choose one of $\{a_i, b_i\}$ for each i and wishes to maximize the number of edges adjacent to the set of chosen vertices.



First we use the probabilistic method to show $N \geq 43$. We select the real number $p = \frac{\sqrt{5}-1}{2} \approx 0.618$ satisfying $p = 1 - p^2$. For each i we then select a_i with probability p and b_i with probability $1 - p$. Then

- Every self-loop (a_i, a_i) is chosen with probability p.
- Any edge (b_i, b_j) is chosen with probability $1 - p^2$.

All other edges are selected with probability at least p, so in expectation we have $68p \approx 42.024$ edges scored. Hence $N \geq 43$.

For a construction showing 43 is optimal, we let $n = 8$, and put five self-loops on each a_i, while taking a single K_8 on the b_i's. The score achieved for selecting m of the a_i's and $8 - m$ of the b_i's is

$$5m + \left(\binom{8}{2} - \binom{m}{2} \right) \leq 43$$

with equality when either $m = 5$ and $m = 6$.

Remark (Colin Tang)**.** Here is one possible motivation for finding the construction. In equality case we probably want all the edges to either be a_i loops or $b_i b_j$ edges. Now if b_i and b_j are not joined by an edge, one can "merge them together", also combining the corresponding a_i's, to get another multigraph with 68 edges whose optimal score is at most the original ones. So by using this smoothing algorithm, we can reduce to a situation where the b_i and b_j are all connected to each other.

It's not unnatural to assume it's a clique then, at which point fiddling with parameters gives the construction. Also, there is a construction for $\lceil 2/3n \rceil$ which is not too difficult to find, and applying this smoothing operation to this construction could suggest a clique of at least 8 vertices too.

Remark (David Lee)**.** One could consider changing the probability $p(n)$ to be a function of the number n of non-loops (hence there are $68 - n$ loops); we would then have

$$\mathbb{E}[\text{points}] \ge (68-n)p(n) + n(1-p(n)^2).$$

The optimal value of $p(n)$ is then

$$p(n) = \begin{cases} \frac{68-n}{2n} = \frac{34}{n} - \frac{1}{2} & n \ge 23 \\ 1 & n < 22. \end{cases}$$

For $n > 23$ we then have

$$\begin{aligned} E(n) =& (68-n)\left(\frac{34}{n} - \frac{1}{2}\right) + n\left(1 - \left(\frac{34}{n} - \frac{1}{2}\right)^2\right) \\ =& \frac{5n}{4} + \frac{34^2}{n} - 34 \end{aligned}$$

which has its worst case at around $5n^2 = 68^2$, so at $n = 30$ and $n = 31$. Indeed, one can find

$$\begin{aligned} E(30) &= 42.033 \\ E(31) &= 42.040. \end{aligned}$$

This gives another way to get the lower bound 43, and gives a hint about approximately how many non-loops one would want in order to achieve such a bound.