USAMO 2003 Solution Notes

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This is an compilation of solutions for the 2003 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

0 Problems 2

1 USAMO 2003/1, proposed by Titu Andreescu 3

2 USAMO 2003/2 4

3 USAMO 2003/3 5

4 USAMO 2003/4, proposed by Titu Andreescu and Zuming Feng 6

5 USAMO 2003/5, proposed by Zuming Feng and Titu Andreescu 7

6 USAMO 2003/6 8

§0 Problems

1. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

2. A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

3. Let n be a positive integer. For every sequence of integers
$$A = (a_0,\ a_1,\ a_2,\ \dots, a_n)$$
satisfying $0 \le a_i \le i$, for $i = 0, \dots, n$, we define another sequence
$$t(A) = (t(a_0),\ t(a_1),\ t(a_2),\ \dots,\ t(a_n))$$
by setting $t(a_i)$ to be the number of terms in the sequence A that precede the term a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

5. Let a, b, c be positive real numbers. Prove that
$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

6. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003^{2003}. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

§1 USAMO 2003/1, proposed by Titu Andreescu

Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

This is immediate by induction on n. For $n = 1$ we take 5; moving forward if M is a working n-digit number then exactly one of

$$\begin{aligned} N_1 &= 10^n + M \\ N_3 &= 3 \cdot 10^n + M \\ N_5 &= 5 \cdot 10^n + M \\ N_7 &= 7 \cdot 10^n + M \\ N_9 &= 9 \cdot 10^n + M \end{aligned}$$

is divisible by 5^{n+1}; as they are all divisible by 5^n and $N_k/5^n$ are all distinct.

§2 USAMO 2003/2

A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

Suppose AB is a side of a polygon in the dissection, lying on diagonal XY, with X, A, B, Y in that order. Then

$$AB = XY - XA - YB.$$

In this way, we see that it actually just suffices to prove the result for a quadrilateral.

To do this, we apply barycentric coordinates. Consider quadrilateral $ABDC$, with $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$. Let $D = (x,y,z)$, with $x+y+z = 1$. By hypothesis, each of the numbers

$$\begin{aligned} -a^2yz + b^2(1-x)z + c^2(1-x)y &= AD^2 \\ a^2(1-y)z + b^2zx + c^2(1-y)x &= BD^2 \\ -a^2(1-z)y - b^2(1-z)x + c^2xy &= CD^2 \end{aligned}$$

is rational. Let $W = a^2yz + b^2zx + c^2xy$. Then,

$$\begin{aligned} b^2z + c^2y &= AD^2 + W \\ a^2z + c^2x &= BD^2 + W \\ a^2y + b^2x &= CD^2 + W. \end{aligned}$$

This implies that $AD^2 + BD^2 + 2W - c^2 = 2S_C z$ and cyclically (as usual $2S_C = a^2 + b^2 - c^2$). If any of S_A, S_B, S_C are zero, then we deduce W is rational. Otherwise, we have that

$$1 = x + y + z = \sum_{\text{cyc}} \frac{AD^2 + BD^2 + 2W - c^2}{2S_C}$$

which implies that W is rational, because it appears with coefficient $\frac{1}{S_A} + \frac{1}{S_B} + \frac{1}{S_C} \neq 0$ (since $S_{BC} + S_{CA} + S_{AB}$ is actually the area of ABC).

Hence from the rationality of W, we deduce that x is rational as long as $S_A \neq 0$, and similarly for the others. So at most one of x, y, z is irrational, but since $x + y + z = 1$ this implies they are all rational.

Finally, if $P = \overline{AD} \cap \overline{BC}$ then $AP = \frac{1}{y+z} AD$, so AP is rational too, completing the proof.

§3 USAMO 2003/3

Let n be a positive integer. For every sequence of integers

$$A = (a_0,\ a_1,\ a_2,\ \dots, a_n)$$

satisfying $0 \le a_i \le i$, for $i = 0, \dots, n$, we define another sequence

$$t(A) = (t(a_0),\ t(a_1),\ t(a_2),\ \dots,\ t(a_n))$$

by setting $t(a_i)$ to be the number of terms in the sequence A that precede the term a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

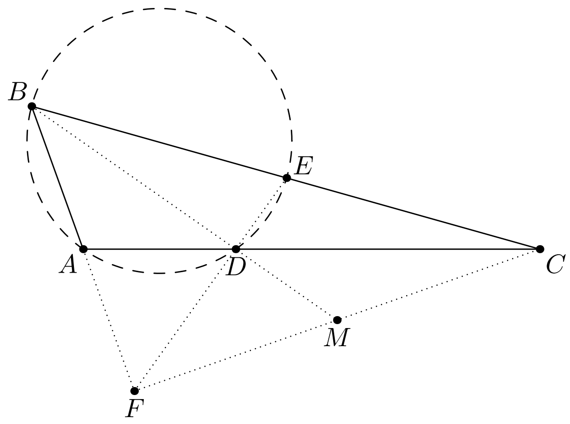
We go by strong induction on n with the base cases $n = 1$ and $n = 2$ done by hand. Consider two cases:

- If $a_0 = 0$ and $a_1 = 1$, then $1 \le t(a_i) \le i$ for $i \ge 1$; now apply induction to
$$(t(a_1) - 1,\ t(a_2) - 1,\ \dots,\ t(a_n) - 1).$$
- Otherwise, assume that $a_0 = a_1 = \dots = a_{k-1} = 0$ but $a_k \ne 0$, where $k \ge 2$. Assume $k < n$ or it's obvious. Then $t(a_i) \ne 0$ for $i \ge k$, thus $t(t(a_i)) \ge k$ for $i \ge k$, and we can apply induction hypothesis to
$$(t(t(a_k)) - k,\ \dots,\ t(t(a_n)) - k).$$

§4 USAMO 2003/4, proposed by Titu Andreescu and Zuming Feng

Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Ceva theorem plus the similar triangles.



We know unconditionally that

$$\measuredangle CBD = \measuredangle EBD = \measuredangle EAD = \measuredangle EAC.$$

Moreover, by Ceva's theorem on $\triangle BCF$, we have $MF = MC \iff \overline{FC} \parallel \overline{AE}$. So we have the equivalences

$$\begin{aligned} MF = MC &\iff \overline{FC} \parallel \overline{AE} \\ &\iff \measuredangle FCA = \measuredangle EAC \\ &\iff \measuredangle MCD = \measuredangle CBD \\ &\iff MC^2 = MB \cdot MD. \end{aligned}$$

§5 USAMO 2003/5, proposed by Zuming Feng and Titu Andreescu

Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

This is a canonical example of tangent line trick. Homogenize so that $a+b+c=3$. The desired inequality reads

$$\sum_{\text{cyc}} \frac{(a+3)^2}{2a^2+(3-a)^2} \le 8.$$

This follows from

$$f(x) = \frac{(x+3)^2}{2x^2+(3-x)^2} \le \frac{1}{3}(4x+4)$$

which can be checked as $\frac{1}{3}(4x+4)(2x^2+(3-x)^2) - (x+3)^2 = (x-1)^2(4x+3) \ge 0$.

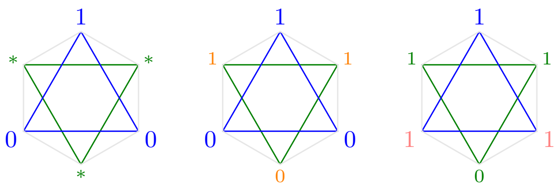
§6 USAMO 2003/6

At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003^{2003}. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

If $a \le b \le c$ are *odd* integers, the configuration which has $(a, b-a, b, c-b, c, c-a)$ around the hexagon in some order (up to cyclic permutation and reflection) is said to be *great* (picture below).

Claim — We can reach a great configuration from any configuration with odd sum.

Proof. We should be able to find an equilateral triangle whose vertices have odd sum. If all three vertices are odd, then we are already done. Otherwise, operate as in the following picture (modulo 2).

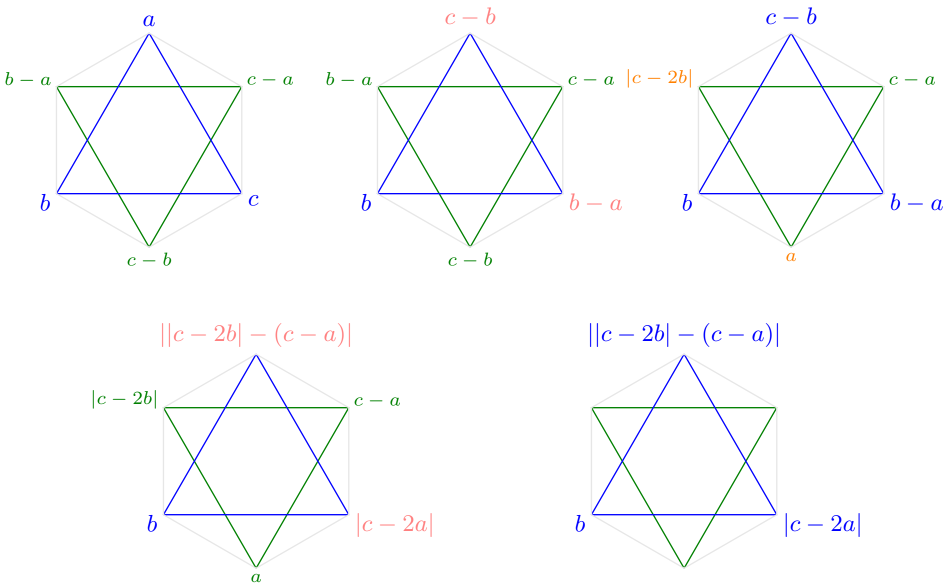


Thus we arrived at a great configuration. $\square$

Claim — Bert's goal is possible for all great configurations.

Proof. If $a = b = c$ then we have $(t, 0, t, 0, t, 0)$ which is obviously winnable.

Otherwise, perform six moves as shown in the diagram to reach a new great configuration whose odd entries are b, $|c-2a|$, $||c-2b|-(c-a)|$ (and perform three more moves to get the even numbers). The idea is to show the largest odd entry has decreased.



This is annoying, but straightforward. Our standing assumption is $a \neq c$ (but possibly $b = c$). It's already obvious that $|c - 2a| < c$, so focus on the last term. If $c > 2b$, then $|(c - 2b) - (c - a)| = |2b - a| < c$ as well for $a \neq c$. When $c \leq 2b$ we instead have $|(2b - c) - (c - a)| \leq \max(2b - c, c - a)$ with equality if and only if $c - a = 0$; but $2b - c \leq c$ as needed. Thus, in all situations we have

$$c \neq a \implies \max\left(\left||c - 2b| - (c - a)\right|, |c - 2a|\right) < c.$$

Now denote the new odd entries by $a' \leq b' \leq c'$ (in some order). If $b < c$ then $c' < c$, while if $b = c$ then $c' = b$ but $b' < c = b$. Thus (c', b', a') precedes (c, b, a) lexicographically, and we can induct down. $\square$

Remark. One simple idea might be to try to overwrite the maximum number at each point, decreasing the sum. However, this fails on the arrangement $(t, t, 0, t, t, 0)$.

Unfortunately, this issue is actually fatal, as the problem has a hidden parity obstruction. The configuration $(1, 1, 0, 1, 1, 0) \bmod 2$ is invariant modulo 2, and so Bert can walk into a "fatal death-trap" of this shape long before the numbers start becoming equal/zero/etc. In other words, you can mess up on the first move! This is why the initial sum is given to be odd; however, it's not possible for Bert to win so one essentially has to "tip-toe" around the 110110 trap any time one leaves the space of odd sum. That's why the great configurations defined above serve as an anchor, making sure we never veer too far from the safe 101010 configuration.

Remark. On the other hand, many other approaches are possible which anchor around a different parity configuration, like 100000 for example. The choice of 101010 by me is due to symmetry — ostensibly, if it worked, there should be fewer cases.