USAMO 1998 Solution Notes

Compiled by Evan Chen

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This is an compilation of solutions for the 1998 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Suppose that the set $\{1, 2, \cdots, 1998\}$ has been partitioned into disjoint pairs $\{a_i, b_i\}$ $(1 \le i \le 999)$ so that for all i, $|a_i - b_i|$ equals 1 or 6. Prove that the sum
$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{999} - b_{999}|$$
ends in the digit 9.

2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and $\mathcal{C}_1$, and let D be the midpoint of $\overline{AB}$. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of $\overline{DE}$ and $\overline{CF}$ intersect at a point M on line AB. Find, with proof, the ratio AM/MC.

3. Let $a_0, a_1, \cdots, a_n$ be numbers from the interval $(0, \pi/2)$ such that $\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \ge n - 1$. Prove that
$$\tan a_0 \tan a_1 \cdots \tan a_n \ge n^{n+1}.$$

4. A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

5. Prove that for each $n \ge 2$, there is a set S of n integers such that $(a - b)^2$ divides ab for every distinct $a, b \in S$.

6. Let $n \ge 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1 A_2 \dots A_n$ for which exactly k of the quadrilaterals $A_i A_{i+1} A_{i+2} A_{i+3}$ have an inscribed circle, where indices are taken modulo n.

§1 USAMO 1998/1

Suppose that the set $\{1, 2, \cdots, 1998\}$ has been partitioned into disjoint pairs $\{a_i, b_i\}$ $(1 \le i \le 999)$ so that for all i, $|a_i - b_i|$ equals 1 or 6. Prove that the sum

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{999} - b_{999}|$$

ends in the digit 9.

Let S be the sum. Modulo 2,

$$S = \sum |a_i - b_i| \equiv \sum (a_i + b_i) = 1 + 2 + \cdots + 1998 \equiv 1 \pmod 2.$$

Modulo 5,

$$S = \sum |a_i - b_i| = 1 \cdot 999 \equiv 4 \pmod 5.$$

So $S \equiv 9 \pmod{10}$.

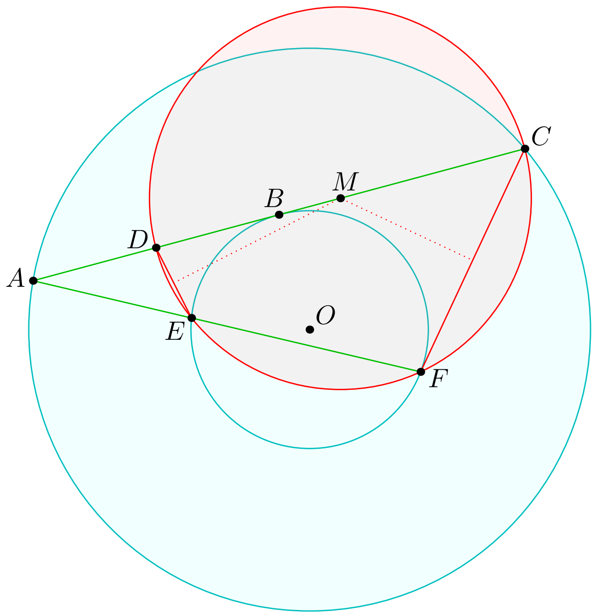
§2 USAMO 1998/2

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and $\mathcal{C}_1$, and let D be the midpoint of $\overline{AB}$. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of $\overline{DE}$ and $\overline{CF}$ intersect at a point M on line AB. Find, with proof, the ratio AM/MC.

By power of a point we have

$$AE \cdot AF = AB^2 = \left(\frac{1}{2}AB\right) \cdot (2AB) = AD \cdot AC$$

and hence $CDEF$ is cyclic. Then M is the circumcenter of quadrilateral $CDEF$.



Thus M is the midpoint of $\overline{CD}$ (and we are given already that B is the midpoint of $\overline{AC}$, D is the midpoint of $\overline{AB}$). Thus a quick computation along $\overline{AC}$ gives $AM/MC = 5/3$.

§3 USAMO 1998/3

Let $a_0, a_1, \cdots, a_n$ be numbers from the interval $(0, \pi/2)$ such that $\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \ge n - 1$. Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \ge n^{n+1}.$$

Let $x_i = \tan(a_i - \frac{\pi}{4})$. Then we have that

$$\tan a_i = \tan(a_i - 45^\circ + 45^\circ) = \frac{x_i + 1}{1 - x_i}.$$

If we further substitute $y_i = \frac{1-x_i}{2} \in (0,1)$, then we have to prove that the following statement:

Claim — If $\sum_0^n y_i \le 1$ and $y_i \ge 0$, we have

$$\prod_{i=1}^{n} \left(\frac{1}{y_i} - 1 \right) \ge n^{n+1}.$$

Proof. Homogenizing, we have to prove that

$$\prod_{i=1}^{n} \left(\frac{y_0 + y_1 + y_2 + \cdots + y_n}{y_i} - 1 \right) \ge n^{n+1}.$$

By AM-GM, we have

$$\frac{y_1 + y_2 + y_3 + \cdots + y_n}{y_0} \ge n \sqrt[n]{\frac{y_1 y_2 y_3 \cdots y_n}{y_1}}.$$

Cyclic product works. □

Remark. Alternatively, the function $x \mapsto \log(1/x - 1)$ is a convex function on $(0,1)$ so Jensen inequality should also work.

§4 USAMO 1998/4

A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

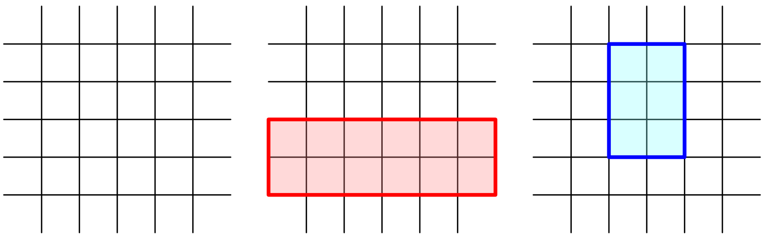
The answer is 98. One of several possible constructions is to toggle all columns and rows with even indices.

In the other direction, let $n = 98$ and suppose that k rectangles are used, none of which are $n \times n$ (else we may delete it). Then, for any two orthogonally adjacent cells, the edge between them must be contained in the edge of one of the k rectangles.

We define a *gridline* to be a line segment that runs in the interior of the board from one side of the board to the other. Hence there are $2n - 2$ gridlines exactly. Moreover, we can classify these rectangles into two types:

- *Full length rectangles*: these span from one edge of the board to the other. The two long sides completely cover two gridlines, but the other two sides of the rectangle do not.
- *Partial length rectangles*: each of four sides can partially cover "half a" gridlines.

See illustration below for $n = 6$.



Since there are $2n - 2$ gridlines; and each rectangle can cover at most two gridlines in total (where partial-length rectangles are "worth $\frac{1}{2}$" on each of the four sides), we immediately get the bound $2k \geq 2n - 2$, or $k \geq n - 1$.

To finish, we prove that:

Claim — If equality holds and $k = n - 1$, then n is odd.

Proof. If equality holds, then look at the horizontal gridlines and say two gridlines are *related* if some rectangle has horizontal edges along both gridlines. (Hence, the graph has degree either 1 or 2 at each vertex, for equality to hold.) The reader may verify the resulting graph consists only of even length cycles and single edges, which would mean $n - 1$ is even. $\square$

Hence for $n = 98$ the answer is indeed 98 as claimed.

§5 USAMO 1998/5

Prove that for each $n \geq 2$, there is a set S of n integers such that $(a-b)^2$ divides ab for every distinct $a, b \in S$.

This is a direct corollary of the more difficult USA TST 2015/2, reproduced below.

> Prove that for every positive integer n, there exists a set S of n positive integers such that for any two distinct $a, b \in S$, $a - b$ divides a and b but none of the other elements of S.

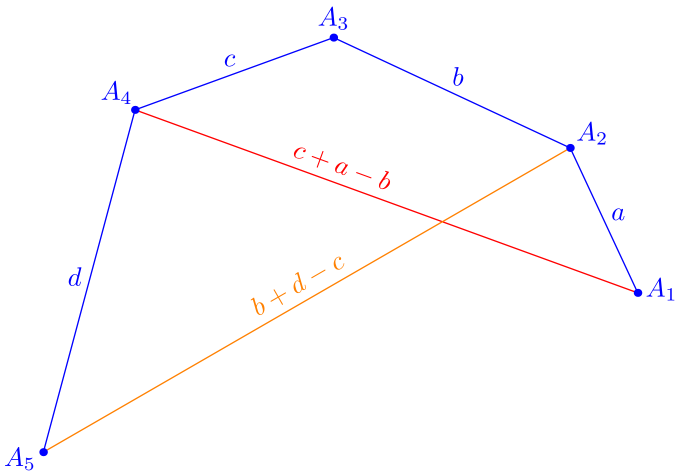
§6 USAMO 1998/6

Let $n \geq 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1A_2\dots A_n$ for which exactly k of the quadrilaterals $A_iA_{i+1}A_{i+2}A_{i+3}$ have an inscribed circle, where indices are taken modulo n.

The main claim is the following:

Claim — We can't have both $A_1A_2A_3A_4$ and $A_2A_3A_4A_5$ be circumscribed.

Proof. If not, then we have the following diagram, where $a = A_1A_2$, $b = A - 2A_3$, $c = A_3A_4$, $d = A_4A_5$.



Then $A_1A_4 = c + a - b$ and $A_5A_2 = b + d - c$. But now

$$A_1A_4 + A_2A_5 = (c + a - b) + (b + d - c) = a + d = A_1A_2 + A_4A_5$$

but in the picture we have an obvious violation of the triangle inequality. □

This immediately gives an upper bound of $\lfloor n/2 \rfloor$.

For the construction, one can construct a suitable cyclic n-gon by using a continuity argument (details to be added).