USAMO 1997 Solution Notes

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This is an compilation of solutions for the 1997 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let $p_1, p_2, p_3, \dots$ be the prime numbers listed in increasing order, and let $0 < x_0 < 1$ be a real number between 0 and 1. For each positive integer k, define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \dfrac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

where $\{x\}$ denotes the fractional part of x. Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \dots$ eventually becomes 0.

2. Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

3. Prove that for any integer n, there exists a unique polynomial Q with coefficients in $\{0, 1, \dots, 9\}$ such that $Q(-2) = Q(-5) = n$.

4. To clip a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon $\mathcal{P}_8$, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is greater than $\frac{1}{3}$, for all $n \geq 6$.

5. If $a, b, c > 0$ prove that

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{1}{abc}.$$

6. Suppose the sequence of nonnegative integers $a_1, a_2, \dots, a_{1997}$ satisfies

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$$

for all $i, j \geq 1$ with $i + j \leq 1997$. Show that there exists a real number x such that $a_n = \lfloor nx \rfloor$ for all $1 \leq n \leq 1997$.

§1 USAMO 1997/1

Let $p_1, p_2, p_3, \dots$ be the prime numbers listed in increasing order, and let $0 < x_0 < 1$ be a real number between 0 and 1. For each positive integer k, define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \dfrac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

where $\{x\}$ denotes the fractional part of x. Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \dots$ eventually becomes 0.

The answer is x_0 rational.

If x_0 is irrational, then all x_i are irrational by induction. So the sequence cannot become zero.

If x_0 is rational, then all are. Now one simply observes that the denominators of x_n are strictly decreasing, until we reach $0 = \frac{0}{1}$. This concludes the proof.

> **Remark.** The sequence p_k could have been any sequence of integers.

§2 USAMO 1997/2

Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

The three lines are the radical axii of the three circles centered at D, E, F, so they concur.

§3 USAMO 1997/3

Prove that for any integer n, there exists a unique polynomial Q with coefficients in $\{0, 1, \dots, 9\}$ such that $Q(-2) = Q(-5) = n$.

If we let

$$Q(x) = \sum_{k \ge 0} a_k x^k$$

then a_k is uniquely determined by $n \pmod{2^k}$ and $n \pmod{5^k}$. Indeed, we can extract the coefficients of Q exactly by the following algorithm:

- Define $b_0 = c_0 = n$.
- For $i \ge 0$, let a_i be the unique digit satisfying $a_i \equiv b_i \pmod 2$, $a_i \equiv c_i \pmod 5$. Then, define

$$b_{i+1} = \frac{b_i - a_i}{-2}, \qquad c_{i+1} = \frac{c_i - a_i}{-5}.$$

The proof is automatic by Chinese remainder theorem, so this shows uniqueness already. The tricky part is to show that all a_i are eventually zero (i.e. the "existence" step is nontrivial because a polynomial may only have finitely many nonzero terms).

In fact, we will prove the following claim:

Claim — Suppose b_0 and c_0 are any integers such that

$$b_0 \equiv c_0 \pmod 3.$$

Then defining b_i and c_i as above, we have $b_i \equiv c_i \pmod 3$ for all i, and $b_N = c_N = 0$ for large enough N.

Proof. Dropping the subscripts for ease of notation, we are looking at the map

$$(b, c) \mapsto \left(\frac{b-a}{-2}, \frac{c-a}{-5} \right)$$

for some $0 \le a \le 9$ (a function in b and c).

The $b \equiv c \pmod 3$ is clearly preserved. Also, examining the size,

- If $|c| > 2$, we have $\left| \frac{c-a}{-5} \right| \le \frac{|c|+9}{5} < |c|$. Thus, we eventually reach a pair with $|c| \le 2$.
- Similarly, if $|b| > 9$, we have $\left| \frac{b-a}{-2} \right| \le \frac{|b|+9}{2} < |b|$, so we eventually reach a pair with $|b| \le 9$.

this leaves us with $5 \cdot 19 = 95$ ordered pairs to check (though only about one third have $b \equiv c \pmod 3$). This can be done by the following code:

```python
import functools
@functools.lru_cache()
def f(x0, y0):
    if x0 == 0 and y0 == 0:
        return 0
    if x0 % 2 == (y0 % 5) % 2:
        d = y0 % 5
```

```
    else:
        d = (y0 % 5) + 5

    x1 = (x0 - d) // (-2)
    y1 = (y0 - d) // (-5)

    return 1 + f(x1, y1)

for x in range(-9, 10):
for y in range(-2, 3):
    if (x % 3 == y % 3):
        print(f"({x:2d}, {y:2d}) finished in {f(x,y)} moves")
```

As this gives the output

```
(-9,  0) finished in 5 moves
(-8, -2) finished in 5 moves
(-8,  1) finished in 5 moves
(-7, -1) finished in 5 moves
(-7,  2) finished in 5 moves
(-6,  0) finished in 3 moves
(-5, -2) finished in 3 moves
(-5,  1) finished in 3 moves
(-4, -1) finished in 3 moves
(-4,  2) finished in 3 moves
(-3,  0) finished in 3 moves
(-2, -2) finished in 3 moves
(-2,  1) finished in 3 moves
(-1, -1) finished in 3 moves
(-1,  2) finished in 3 moves
( 0,  0) finished in 0 moves
( 1, -2) finished in 2 moves
( 1,  1) finished in 1 moves
( 2, -1) finished in 2 moves
( 2,  2) finished in 1 moves
( 3,  0) finished in 2 moves
( 4, -2) finished in 2 moves
( 4,  1) finished in 2 moves
( 5, -1) finished in 2 moves
( 5,  2) finished in 2 moves
( 6,  0) finished in 4 moves
( 7, -2) finished in 4 moves
( 7,  1) finished in 4 moves
( 8, -1) finished in 4 moves
( 8,  2) finished in 4 moves
( 9,  0) finished in 4 moves
```

we are done. □

§4 USAMO 1997/4

To clip a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon $\mathcal{P}_8$, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is greater than $\frac{1}{3}$, for all $n \ge 6$.

Call the original hexagon $ABCDEF$. We show the area common to triangles ACE and BDF is in every $\mathcal{P}_n$; this solves the problem since the area is $1/3$.

For every side of a clipped polygon, we define its *foundation* recursively as follows:

- AB, BC, CD, DE, EF, FA are each their own foundation (we also call these *original sides*).
- When a new clipped edge is added, its foundation is the union of the foundations of the two edges it touches.

Hence, any foundations are nonempty subsets of original sides.

Claim — All foundations are in fact at most two-element sets of adjacent original sides.

Proof. It's immediate by induction that any two adjacent sides have at most two elements in the union of their foundations, and if there are two, they are two adjacent original sides. □

Now, if a side has foundation contained in $\{AB, BC\}$, say, then the side should be contained within triangle ABC. Hence the side does not touch AC. This proves the problem.

§5 USAMO 1997/5

If $a, b, c > 0$ prove that

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \le \frac{1}{abc}.$$

From $a^3 + b^3 \ge ab(a+b)$, the left-hand side becomes

$$\sum_{\text{cyc}} \frac{1}{a^3+b^3+abc} \le \sum_{\text{cyc}} \frac{1}{ab(a+b+c)} = \frac{1}{abc} \sum_{\text{cyc}} \frac{c}{a+b+c} = \frac{1}{abc}.$$

§6 USAMO 1997/6

Suppose the sequence of nonnegative integers $a_1, a_2, \dots, a_{1997}$ satisfies

$$a_i + a_j \le a_{i+j} \le a_i + a_j + 1$$

for all $i, j \ge 1$ with $i + j \le 1997$. Show that there exists a real number x such that $a_n = \lfloor nx \rfloor$ for all $1 \le n \le 1997$.

We are trying to show there exists an $x \in \mathbb{R}$ such that

$$\frac{a_n}{n} \le x < \frac{a_n + 1}{n} \qquad \forall n.$$

This means we need to show

$$\max_i \frac{a_i}{i} < \min_j \frac{a_j + 1}{j}.$$

Replace 1997 by N. We will prove this by induction, but we will need some extra hypotheses on the indices i, j which are used above.

Claim — Suppose that

- Integers $a_1, a_2, \dots, a_N$ satisfy the given conditions.
- Let $i = \operatorname{argmax}_n \frac{a_n}{n}$; if there are ties, pick the smallest i.
- Let $j = \operatorname{argmin}_n \frac{a_n+1}{n}$; if there are ties, pick the smallest j.

Then

$$\frac{a_i}{i} < \frac{a_j + 1}{j}.$$

Moreover, these two fractions are in lowest terms, and are adjacent in the Farey sequence of order $\max(i, j)$.

Proof. By induction on $N \ge 1$ with the base case clear. So suppose we have the induction hypothesis with numbers $a_1, \dots, a_{N-1}$, with i and j as promised.

Now, consider the new number a_N. We have two cases:

- Suppose $i + j > N$. Then, no fraction with denominator N can lie strictly inside the interval; so we may write for some integer b

$$\frac{b}{N} \le \frac{a_i}{i} < \frac{a_j + 1}{j} \le \frac{b+1}{N}.$$

For purely algebraic reasons we have

$$\frac{b - a_i}{N - i} \le \frac{b}{N} \le \frac{a_i}{i} < \frac{a_j + 1}{j} \le \frac{b+1}{N} \le \frac{b - a_j}{N - j}.$$

Now,

$$\begin{aligned} a_N &\ge a_i + a_{N-i} \ge a_i + (N - i) \cdot \frac{a_i}{i} \\ &\ge a_i + (b - a_i) = b \\ a_N &\le a_j + a_{N-j} + 1 \le (a_j + 1) + (N - j) \cdot \frac{a_j + 1}{j} \\ &= (a_j + 1) + (b - a_j) = b + 1. \end{aligned}$$

Thus $a_N \in \{b, b+1\}$. This proves that $\frac{a_N}{N} \le \frac{a_i}{i}$ while $\frac{a_N+1}{N} \ge \frac{a_j+1}{j}$. Moreover, the pair (i, j) does not change, so all inductive hypotheses carry over.

- On the other hand, suppose $i + j = N$. Then we have

$$\frac{a_i}{i} < \frac{a_i + a_j + 1}{N} < \frac{a_j + 1}{j}.$$

Now, we know a_N could be either $a_i + a_j$ or $a_i + a_j + 1$. If it's the former, then (i, j) becomes (i, N). If it's the latter, then (i, j) becomes (N, j). The properties of Farey sequences ensure that the $\frac{a_i+a_j+1}{N}$ is reduced, either way.

□