USA TST Selection Test for 66th IMO and 14th EGMO

Pittsburgh, PA

Day I   1:20pm – 5:50pm

Tuesday, June 18, 2024

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

You may keep the problems, but you cannot discuss them publicly until they are posted by staff online.

Problem 1. For every ordered pair of integers (i, j), not necessarily positive, we wish to select a point $P_{i,j}$ in the Cartesian plane whose coordinates lie inside the unit square defined by

$$i < x < i + 1, \qquad j < y < j + 1.$$

Find all real numbers $c > 0$ for which it's possible to choose these points such that for all integers i and j, the (possibly concave or degenerate) quadrilateral $P_{i,j}P_{i+1,j}P_{i+1,j+1}P_{i,j+1}$ has perimeter strictly less than c.

Problem 2. Let p be an odd prime number. Suppose P and Q are polynomials with integer coefficients such that $P(0) = Q(0) = 1$, there is no nonconstant polynomial dividing both P and Q, and

$$1 + \cfrac{x}{1 + \cfrac{2x}{\ddots \, \cfrac{}{1 + \cfrac{}{1 + (p-1)x}}}} = \frac{P(x)}{Q(x)}.$$

Show that all coefficients of P except for the constant coefficient are divisible by p, and all coefficients of Q are *not* divisible by p.

Problem 3. Let $A = \{a_1, \dots, a_{2024}\}$ be a set of 2024 pairwise distinct real numbers. Assume that there exist positive integers $b_1, b_2, \dots, b_{2024}$ such that

$$a_1b_1 + a_2b_2 + \cdots + a_{2024}b_{2024} = 0.$$

Prove that one can choose $a_{2025}, a_{2026}, a_{2027}, \dots$ such that $a_k \in A$ for all $k \geq 2025$ and, for every positive integer d, there exist infinitely many positive integers n satisfying

$$\sum_{k=1}^{n} a_k k^d = 0.$$

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Day II 1:20pm – 5:50pm

Thursday, June 20, 2024

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

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Problem 4. Let $ABCD$ be a quadrilateral inscribed in a circle with center O and E be the intersection of segments AC and BD. Let ω_1 be the circumcircle of ADE and ω_2 be the circumcircle of BCE. The tangent to ω_1 at A and the tangent to ω_2 at C meet at P. The tangent to ω_1 at D and the tangent to ω_2 at B meet at Q. Show that $OP = OQ$.

Problem 5. For a positive integer k, let $s(k)$ denote the number of 1s in the binary representation of k. Prove that for any positive integer n,

$$\sum_{i=1}^{n}(-1)^{s(3i)} > 0.$$

Problem 6. Determine whether there exists a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for all positive integers m and n,

$$f(m + nf(m)) = f(n)^m + 2024! \cdot m.$$

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Day III 1:20pm – 5:50pm

Saturday, June 22, 2024

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

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Problem 7. An infinite sequence a_1, a_2, a_3, $\ldots$ of real numbers satisfies

$$a_{2n-1} + a_{2n} > a_{2n+1} + a_{2n+2} \qquad \text{and} \qquad a_{2n} + a_{2n+1} < a_{2n+2} + a_{2n+3}$$

for every positive integer n. Prove that there exists a real number C such that $a_n a_{n+1} < C$ for every positive integer n.

Problem 8. Let ABC be a scalene triangle, and let D be a point on side BC satisfying $\angle BAD = \angle DAC$. Suppose that X and Y are points inside ABC such that triangles ABX and ACY are similar and quadrilaterals $ACDX$ and $ABDY$ are cyclic. Let lines BX and CY meet at S and lines BY and CX meet at T. Prove that lines DS and AT are parallel.

Problem 9. Let $n \geq 2$ be a fixed integer. The cells of an $n \times n$ table are filled with the integers from 1 to n^2 with each number appearing exactly once. Let N be the number of unordered quadruples of cells on this board which form an axis-aligned rectangle, with the two smaller integers being on opposite vertices of this rectangle. Find the largest possible value of N.