USA TST Selection Test for 65$^{\text{th}}$ IMO and 13$^{\text{th}}$ EGMO

Pittsburgh, PA

Day I 1:15pm – 5:45pm

Tuesday, June 20, 2023

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 1. Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that

$$\angle ABS = \angle ACR = 180^\circ - \angle BGC.$$

Prove that $\angle RAS + \angle BAC = \angle BGC$.

Problem 2. Let $n \geq m \geq 1$ be integers. Prove that

$$\sum_{k=m}^{n} \left(\frac{1}{k^2} + \frac{1}{k^3} \right) \geq m \cdot \left(\sum_{k=m}^{n} \frac{1}{k^2} \right)^2 .$$

Problem 3. Find all positive integers n for which it is possible to color some cells of an infinite grid of unit squares red, such that each rectangle consisting of exactly n cells (and whose edges lie along the lines of the grid) contains an odd number of red cells.

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Day II   1:15pm – 5:45pm

Thursday, June 22, 2023

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

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Problem 4. Let $n \geq 3$ be an integer and let K_n be the complete graph on n vertices. Each edge of K_n is colored either red, green, or blue. Let A denote the number of triangles in K_n with all edges of the same color, and let B denote the number of triangles in K_n with all edges of different colors. Prove that

$$B \leq 2A + \frac{n(n-1)}{3}.$$

(The *complete graph* on n vertices is the graph on n vertices with $\binom{n}{2}$ edges, with exactly one edge joining every pair of vertices. A *triangle* consists of the set of $\binom{3}{2} = 3$ edges between 3 of these n vertices.)

Problem 5. Suppose a, b, and c are three complex numbers with product 1. Assume that none of a, b, and c are real or have absolute value 1. Define

$$p = (a+b+c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \qquad \text{and} \qquad q = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

Given that both p and q are real numbers, find all possible values of the ordered pair (p, q).

Problem 6. Let ABC be a scalene triangle and let P and Q be two distinct points in its interior. Suppose that the angle bisectors of $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$ are the altitudes of triangle ABC. Prove that the midpoint of $\overline{PQ}$ lies on the Euler line of ABC.

(The Euler line is the line through the circumcenter and orthocenter of a triangle.)

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Day III 1:15pm – 5:45pm

Saturday, June 24, 2023

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 7. The Bank of Pittsburgh issues coins that have a heads side and a tails side. Vera has a row of 2023 such coins alternately tails-up and heads-up, with the leftmost coin tails-up.

In a *move*, Vera may flip over one of the coins in the row, subject to the following rules:

- On the first move, Vera may flip over any of the 2023 coins.
- On all subsequent moves, Vera may only flip over a coin adjacent to the coin she flipped on the previous move. (We do not consider a coin to be adjacent to itself.)

Determine the smallest possible number of moves Vera can make to reach a state in which every coin is heads-up.

Problem 8. Let ABC be an equilateral triangle with side length 1. Points A_1 and A_2 are chosen on side BC, points B_1 and B_2 are chosen on side CA, and points C_1 and C_2 are chosen on side AB such that $BA_1 < BA_2$, $CB_1 < CB_2$, and $AC_1 < AC_2$.

Suppose that the three line segments B_1C_2, C_1A_2, and A_1B_2 are concurrent, and the perimeters of triangles AB_2C_1, BC_2A_1, and CA_2B_1 are all equal. Find all possible values of this common perimeter.

Problem 9. For every integer $m \geq 1$, let $\mathbb{Z}/m\mathbb{Z}$ denote the set of integers modulo m.

Let p be a fixed prime and let $a \geq 2$ and $e \geq 1$ be fixed integers. Given a function $f\colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ and an integer $k \geq 0$, the *kth finite difference*, denoted $\Delta^k f$, is the function from $\mathbb{Z}/a\mathbb{Z}$ to $\mathbb{Z}/p^e\mathbb{Z}$ defined recursively by

$$\begin{aligned} \Delta^0 f(n) &= f(n) \\ \Delta^k f(n) &= \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n) \qquad \text{for } k = 1, 2, \ldots. \end{aligned}$$

Determine the number of functions f such that there exists some $k \geq 1$ for which $\Delta^k f = f$.