USA TST Selection Test for 64$^{\text{th}}$ IMO and 12$^{\text{th}}$ EGMO

Pittsburgh, PA

Day I 1:15pm – 5:45pm

Tuesday, June 21, 2022

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 1. Let n be a positive integer. Find the smallest positive integer k such that for any set S of n points in the interior of the unit square, there exists a set of k rectangles such that the following hold:

- The sides of each rectangle are parallel to the sides of the unit square.
- Each point in S is *not* in the interior of any rectangle.
- Each point in the interior of the unit square but *not* in S is in the interior of at least one of the k rectangles.

(The interior of a polygon does not contain its boundary.)

Problem 2. Let ABC be a triangle. Let θ be a fixed angle for which

$$\theta < \frac{1}{2}\min(\angle A, \angle B, \angle C).$$

Points S_A and T_A lie on segment BC such that $\angle BAS_A = \angle T_AAC = \theta$. Let P_A and Q_A be the feet from B and C to $\overline{AS_A}$ and $\overline{AT_A}$ respectively. Then ℓ_A is defined as the perpendicular bisector of $\overline{P_AQ_A}$.

Define ℓ_B and ℓ_C analogously by repeating this construction two more times (using the same value of θ). Prove that ℓ_A, ℓ_B, and ℓ_C are concurrent or all parallel.

Problem 3. Determine all positive integers N for which there exists a strictly increasing sequence of positive integers $s_0 < s_1 < s_2 < \cdots$ satisfying the following properties:

- the sequence $s_1 - s_0,\ s_2 - s_1,\ s_3 - s_2, \ldots$ is periodic; and
- $s_{s_n} - s_{s_{n-1}} \le N < s_{1+s_n} - s_{s_{n-1}}$ for all positive integers n.

USA TST Selection Test for 64^{th} IMO and 12^{th} EGMO

Pittsburgh, PA

Day II 1:15pm – 5:45pm

Thursday, June 23, 2022

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 4. Let $\mathbb{N}$ denote the set of positive integers. A function $f : \mathbb{N} \to \mathbb{N}$ has the property that for all positive integers m and n, exactly one of the $f(n)$ numbers

$$f(m+1), f(m+2), \ldots, f(m+f(n))$$

is divisible by n. Prove that $f(n) = n$ for infinitely many positive integers n.

Problem 5. Let $A_1, \ldots, A_{2022}$ be the vertices of a regular 2022-gon in the plane. Alice and Bob play a game. Alice secretly chooses a line and colors all points in the plane on one side of the line blue, and all points on the other side of the line red. Points on the line are colored blue, so every point in the plane is either red or blue. (Bob cannot see the colors of the points.)

In each round, Bob chooses a point in the plane (not necessarily among $A_1, \ldots, A_{2022}$) and Alice responds truthfully with the color of that point. What is the smallest number Q for which Bob has a strategy to always determine the colors of points $A_1, \ldots, A_{2022}$ in Q rounds?

Problem 6. Let O and H be the circumcenter and orthocenter, respectively, of an acute scalene triangle ABC. The perpendicular bisector of $\overline{AH}$ intersects $\overline{AB}$ and $\overline{AC}$ at X_A and Y_A respectively. Let K_A denote the intersection of the circumcircles of triangles OX_AY_A and BOC other than O.

Define K_B and K_C analogously by repeating this construction two more times. Prove that K_A, K_B, K_C, and O are concyclic.

USA TST Selection Test for 64$^{\text{th}}$ IMO and 12$^{\text{th}}$ EGMO

Pittsburgh, PA

Day III 1:15pm – 5:45pm

Saturday, June 25, 2022

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 7. Let $ABCD$ be a parallelogram. Point E lies on segment CD such that

$$2\angle AEB = \angle ADB + \angle ACB,$$

and point F lies on segment BC such that

$$2\angle DFA = \angle DCA + \angle DBA.$$

Let K be the circumcenter of triangle ABD. Prove that $KE = KF$.

Problem 8. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{Z}$ such that

$$\left\lfloor \frac{f(mn)}{n} \right\rfloor = f(m)$$

for all positive integers m, n.

Problem 9. Let $k > 1$ be a fixed positive integer. Prove that if n is a sufficiently large positive integer, there exists a sequence of integers with the following properties:

- Each element of the sequence is between 1 and n, inclusive.
- For any two different contiguous subsequences of the sequence with length between 2 and k inclusive, the multisets of values in those two subsequences is not the same.
- The sequence has length at least $0.499n^2$.