USA Team Selection Test for 62$^{\text{nd}}$ IMO and 10$^{\text{th}}$ EGMO

United States of America

Day I

November 12, 2020

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 1. Let a, b, c be fixed positive integers. There are $a + b + c$ ducks sitting in a circle, one behind the other. Each duck picks either *rock*, *paper*, or *scissors*, with a ducks picking rock, b ducks picking paper, and c ducks picking scissors.

A *move* consists of an operation of one of the following three forms:

- If a duck picking rock sits behind a duck picking scissors, they switch places.
- If a duck picking paper sits behind a duck picking rock, they switch places.
- If a duck picking scissors sits behind a duck picking paper, they switch places.

Determine, in terms of a, b, and c, the maximum number of moves which could take place, over all possible initial configurations.

Problem 2. Let ABC be a scalene triangle with incenter I. The incircle of ABC touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at points D, E, F, respectively. Let P be the foot of the altitude from D to $\overline{EF}$, and let M be the midpoint of $\overline{BC}$. The rays AP and IP intersect the circumcircle of triangle ABC again at points G and Q, respectively. Show that the incenter of triangle GQM coincides with D.

Problem 3. We say a nondegenerate triangle whose angles have measures θ_1, θ_2, θ_3 is *quirky* if there exist integers r_1, r_2, r_3, not all zero, such that

$$r_1\theta_1 + r_2\theta_2 + r_3\theta_3 = 0.$$

Find all integers $n \geq 3$ for which a triangle with side lengths $n - 1$, n, $n + 1$ is quirky.

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Day II

December 10, 2020

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Problem 4. Find all pairs of positive integers (a, b) satisfying the following conditions:

(i) a divides $b^4 + 1$,

(ii) b divides $a^4 + 1$,

(iii) $\lfloor \sqrt{a} \rfloor = \lfloor \sqrt{b} \rfloor$.

Problem 5. Let $\mathbb{N}^2$ denote the set of ordered pairs of positive integers. A finite subset S of $\mathbb{N}^2$ is *stable* if whenever (x, y) is in S, then so are all points (x', y') of $\mathbb{N}^2$ with both $x' \leq x$ and $y' \leq y$.

Prove that if S is a stable set, then among all stable subsets of S (including the empty set and S itself), at least half of them have an even number of elements.

Problem 6. Let A, B, C, D be four points such that no three are collinear and D is not the orthocenter of triangle ABC. Let P, Q, R be the orthocenters of $\triangle BCD$, $\triangle CAD$, $\triangle ABD$, respectively. Suppose that lines AP, BQ, CR are pairwise distinct and are concurrent. Show that the four points A, B, C, D lie on a circle.

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Day III

January 21, 2021

Time limit: 4.5 hours. You may keep the problems, but they should not be posted until next Monday at noon Eastern time.

Problem 7. Find all nonconstant polynomials $P(z)$ with complex coefficients for which all complex roots of the polynomials $P(z)$ and $P(z)-1$ have absolute value 1.

Problem 8. For every positive integer N, let $\sigma(N)$ denote the sum of the positive integer divisors of N. Find all integers $m \geq n \geq 2$ satisfying

$$\frac{\sigma(m)-1}{m-1} = \frac{\sigma(n)-1}{n-1} = \frac{\sigma(mn)-1}{mn-1}.$$

Problem 9. Ten million fireflies are glowing in $\mathbb{R}^3$ at midnight. Some of the fireflies are friends, and friendship is always mutual. Every second, one firefly moves to a new position so that its distance from each one of its friends is the same as it was before moving. This is the only way that the fireflies ever change their positions. No two fireflies may ever occupy the same point.

Initially, no two fireflies, friends or not, are more than a meter away. Following some finite number of seconds, all fireflies find themselves at least ten million meters away from their original positions. Given this information, find the greatest possible number of friendships between the fireflies.