USA TST Selection Test for 61^{st} IMO and 9^{th} EGMO

Pittsburgh, PA

Day I 1:15pm – 5:45pm

Tuesday, June 18, 2019

Problem 1. Find all binary operations $\diamondsuit\colon \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning $\diamondsuit$ takes pairs of positive real numbers to positive real numbers) such that for any real numbers $a, b, c > 0$,

- the equation $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$ holds; and
- if $a \geq 1$ then $a \diamondsuit a \geq 1$.

Problem 2. Let ABC be an acute triangle with circumcircle Ω and orthocenter H. Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q, respectively. Denote by ω the circumcircle of $\triangle ADE$.

(a) Show that lines PE and QD meet on ω.

(b) Prove that if ω passes through H, then lines PD and QE meet on ω as well.

Problem 3. On an infinite square grid we place finitely many *cars*, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that the cell immediately in front of each car is empty, and moreover no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.). In a *move*, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times.

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Day II 1:15pm – 5:45pm

Thursday, June 20, 2019

Problem 4. Consider coins with positive real denominations not exceeding 1. Find the smallest $C > 0$ such that the following holds: if we are given any 100 such coins with total value 50, then we can always split them into two stacks of 50 coins each such that the absolute difference between the total values of the two stacks is at most C.

Problem 5. Let ABC be an acute triangle with orthocenter H and circumcircle Γ. A line through H intersects segments AB and AC at E and F, respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ again at a point D. Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ.

Problem 6. Suppose P is a polynomial with integer coefficients such that for every positive integer n, the sum of the decimal digits of $|P(n)|$ is not a Fibonacci number. Must P be constant?

(A *Fibonacci number* is an element of the sequence F_0, F_1, $\ldots$ defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$.)

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Day III 1:15pm – 5:45pm

Saturday, June 22, 2019

Problem 7. Let $f\colon \mathbb{Z} \to \{1, 2, \dots, 10^{100}\}$ be a function satisfying

$$\gcd(f(x), f(y)) = \gcd(f(x), x - y)$$

for all integers x and y. Show that there exist positive integers m and n such that $f(x) = \gcd(m + x, n)$ for all integers x.

Problem 8. Let $\mathcal{S}$ be a set of 16 points in the plane, no three collinear. Let $\chi(\mathcal{S})$ denote the number of ways to draw 8 line segments with endpoints in $\mathcal{S}$, such that no two drawn segments intersect, even at endpoints. Find the smallest possible value of $\chi(\mathcal{S})$ across all such $\mathcal{S}$.

Problem 9. Let ABC be a triangle with incenter I. Points K and L are chosen on segment BC such that the incircles of $\triangle ABK$ and $\triangle ABL$ are tangent at P, and the incircles of $\triangle ACK$ and $\triangle ACL$ are tangent at Q. Prove that $IP = IQ$.