

**2017 USA Team Selection Test Selection Test Day 1**  
**Carnegie Mellon University**  
**June 24, 2017**  
**1:15 – 5:45pm**

1. Let  $ABC$  be a triangle with circumcircle  $\Gamma$ , circumcenter  $O$ , and orthocenter  $H$ . Assume that  $AB \neq AC$ . Let  $M$  and  $N$  be the midpoints of sides  $AB$  and  $AC$ , respectively, and let  $E$  and  $F$  be the feet of the altitudes from  $B$  and  $C$  in  $\triangle ABC$ , respectively. Let  $P$  be the intersection point of line  $MN$  with the tangent line to  $\Gamma$  at  $A$ . Let  $Q$  be the intersection point, other than  $A$ , of  $\Gamma$  with the circumcircle of  $\triangle AEF$ . Let  $R$  be the intersection point of lines  $AQ$  and  $EF$ . Prove that  $PR \perp OH$ .
2. Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. (The word does not need to be a valid English word.) Then Banana picks a nonnegative integer  $k$  and challenges Ana to supply a word with exactly  $k$  subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses.

For example, if Ana picks the word "TST", and Banana chooses  $k = 4$ , then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word.

Which words can Ana pick so that she can win no matter what value of  $k$  Banana chooses?

(The subsequences of a string of length  $n$  are the  $2^n$  strings which are formed by deleting some of its characters, possibly all or none, while preserving the order of the remaining characters.)

3. Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)},$$

where  $f$  and  $g$  are polynomials with nonnegative real coefficients. For each  $c > 0$ , determine the minimum possible degree of  $f$ , or show that no such  $f, g$  exist.

The only allowed materials are ruler, compass, writing utensils, blank paper. Protractors and graph paper are not permitted. Write your TSTST ID, page number, and full problem number on all pages. **Do not write your name.**

**2017 USA Team Selection Test Selection Test Day 2**  
**Carnegie Mellon University**  
**June 26, 2017**  
**1:15 – 5:45pm**

4. Find all nonnegative integer solutions to  $2^a + 3^b + 5^c = n!$ .
5. Let  $ABC$  be a triangle with incenter  $I$ . Let  $D$  be a point on side  $BC$  and let  $\omega_B$  and  $\omega_C$  be the incircles of  $\triangle ABD$  and  $\triangle ACD$ , respectively. Suppose that  $\omega_B$  and  $\omega_C$  are tangent to segment  $BC$  at points  $E$  and  $F$ , respectively. Let  $P$  be the intersection of segment  $AD$  with the line joining the centers of  $\omega_B$  and  $\omega_C$ . Let  $X$  be the intersection point of lines  $BI$  and  $CP$  and let  $Y$  be the intersection point of lines  $CI$  and  $BP$ . Prove that lines  $EX$  and  $FY$  meet on the incircle of  $\triangle ABC$ .
6. A sequence of positive integers  $(a_n)_{n \geq 1}$  is of *Fibonacci type* if it satisfies the recursive relation  $a_{n+2} = a_{n+1} + a_n$  for all  $n \geq 1$ . Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

The only allowed materials are ruler, compass, writing utensils, blank paper. Protractors and graph paper are not permitted. Write your TSTST ID, page number, and full problem number on all pages. **Do not write your name.**