USA TST Selection Test for 59$^{\text{th}}$ IMO and 7$^{\text{th}}$ EGMO

Pittsburgh, PA

Day I 1:15pm – 5:45pm

Saturday, June 24, 2017

Problem 1. Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and $\angle A \neq 90^\circ$. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection point of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection point of lines AQ and EF. Prove that $\overline{PR} \perp \overline{OH}$.

Problem 2. Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

Problem 3. Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)}$$

where f and g are nonzero polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.

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Day II   1:15pm – 5:45pm

Monday, June 26, 2017

Problem 4. Find all nonnegative integer solutions to $2^a + 3^b + 5^c = n!$.

Problem 5. Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

Problem 6. A sequence of positive integers $(a_n)_{n\geq 1}$ is of *Fibonacci type* if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?