1. Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers, and let $m$ be a fixed positive integer less than $n$. We say an index $k$ with $1 \leq k \leq n$ is good if there exists some $\ell$ with $1 \leq \ell \leq m$ such that

$$a_k + a_{k+1} + \cdots + a_{k+\ell-1} \geq 0,$$

where the indices are taken modulo $n$. Let $T$ be the set of all good indices. Prove that

$$\sum_{k \in T} a_k \geq 0.$$ 

2. Let $ABC$ be a scalene triangle. Let $K_a$, $L_a$, and $M_a$ be the respective intersections with $BC$ of the internal angle bisector, external angle bisector, and the median from $A$. The circumcircle of $AK_aL_a$ intersects $AM_a$ a second time at a point $X_a$ different from $A$. Define $X_b$ and $X_c$ analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of $ABC$.

(The Euler line of $ABC$ is the line passing through the circumcenter, centroid, and orthocenter of $ABC$.)

3. Let $P$ be the set of all primes, and let $M$ be a non-empty subset of $P$. Suppose that for any non-empty subset $\{p_1, p_2, \ldots, p_k\}$ of $M$, all prime factors of $p_1p_2\cdots p_k + 1$ are also in $M$. Prove that $M = P$. 
4. Let $x$, $y$, and $z$ be real numbers (not necessarily positive) such that $x^4 + y^4 + z^4 + xyz = 4$. Show that

$$x \leq 2 \quad \text{and} \quad \sqrt{2 - x} \geq \frac{y + z}{2}.$$ 

5. Let $\varphi(n)$ denote the number of positive integers less than $n$ that are relatively prime to $n$. Prove that there exists a positive integer $m$ for which the equation $\varphi(n) = m$ has at least 2015 solutions in $n$.

6. A Nim-style game is defined as follows. Two positive integers $k$ and $n$ are specified, along with a finite set $S$ of $k$-tuples of integers (not necessarily positive). At the start of the game, the $k$-tuple $(n, 0, 0, \ldots, 0)$ is written on the blackboard.

A legal move consists of erasing the tuple $(a_1, a_2, \ldots, a_k)$ which is written on the blackboard and replacing it with $(a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k)$, where $(b_1, b_2, \ldots, b_k)$ is an element of the set $S$. Two players take turns making legal moves, and the first to write a negative integer loses. In the event that neither player is ever forced to write a negative integer, the game is a draw.

Prove that there is a choice of $k$ and $S$ with the following property: the first player has a winning strategy if $n$ is a power of 2, and otherwise the second player has a winning strategy.