Solutions to TSTST 2015

United States of America

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§1 Solution to TSTST Problem 1

This problem was proposed by Mark Sellke.

Define

$$M(k) = \max_{1 \le \ell \le m} (a_k + a_{k+1} + \cdots + a_{k+\ell-1}),$$

and let $N(k) = \max(M(k), 0)$. Note that $k \in T \iff M(k) = N(k)$. For fixed $k_0 \in T$, take $\ell_0 \ge 1$ achieving the maximum value for $M(k_0)$, i.e. with

$$M(k_0) = (a_{k_0} + \cdots + a_{k_0+\ell_0-1}).$$

We may rewrite this as

$$a_{k_0} = M(k_0) - (a_{k_0+1} + \cdots + a_{k_0+\ell_0-1}).$$

From the definition of N we have now

$$a_{k_0} = M(k_0) - (a_{k_0+1} + \cdots + a_{k_0+\ell_0-1}) \ge M(k_0) - N(k_0+1) = N(k_0) - N(k_0+1).$$

Now, divide the multi-set of values $N(k)$ for $1 \le k \le n$ into the following three subsets: the **positive**, **zero**, and **negative** values. The values $N(k)$ for $k \in T$ consist of all positive values, some 0 values, and no negative values. This means that among all sets $S \subseteq \{1, 2, \ldots n\}$ with $|S| = |T|$, $\sum_{k \in S} N(k)$ is maximized (perhaps not uniquely) when $T = S$. In particular,

$$\sum_{k \in T} N(k) \ge \sum_{k \in T} N(k+1).$$

From this, we conclude the problem statement, as

$$\sum_{k \in T} a_k \ge \sum_{k \in T} (N(k) - N(k+1)) \ge 0.$$

§2 Solution to TSTST Problem 2

This problem was proposed by Ivan Borsenco.

Step 1. Let $BC = a$, $AC = b$, $AB = c$. Then the bisector theorem gives $K_aB/K_aC = L_aB/L_aC = c/b$. Using the fact that $K_aB + K_aC = a$ and $|L_aB - L_aC| = a$, we obtain

$$K_aB = a\frac{ca}{b+c}, \quad L_aB = a\frac{ca}{|b-c|}.$$

Then

$$M_aK_a = \frac{a}{2}\frac{|b-c|}{b+c}, \quad M_aL_a = \frac{a}{2}\frac{b+c}{|b-c|}.$$

Note also that K_a and L_a are on the same side of BC. Using the power of M_a with respect to the circumcircle of AK_aL_a we obtain

$$M_aX_a \cdot M_aA = M_aK_a \cdot M_aL_a = \frac{a^2}{4} = M_aB^2 = M_aC^2.$$

The first step can be proved differently as follows. Because X_a lies on Apollonius circle corresponding to vertex A, we have $\frac{BX_a}{CX_a} = \frac{AB}{AC}$. From Stewart's formulae we have

$$\begin{aligned}\frac{AB^2}{AC^2} = \frac{BX_a^2}{CX_a^2} &= \frac{AB^2 \cdot X_aM_a + BM_a^2 \cdot AX_a - AX_a \cdot X_aM_a \cdot AM_a}{AC^2 \cdot X_aM_a + CM_a^2 \cdot AX_a - AX_a \cdot X_aM_a \cdot AM_a} \\ &= \frac{BM_a^2 \cdot AX_a - AX_a \cdot X_aM_a \cdot AM_a}{CM_a^2 \cdot AX_a - AX_a \cdot X_aM_a \cdot AM_a},\end{aligned}$$

which is possible if and only if $BM_a^2 = CM_a^2 = X_aM_a \cdot AM_a$.

Step 2. By step 1 triangles M_aBA and M_aX_aB are similar, and triangles M_aCA and M_aX_aC are similar. Consequently

$$\angle BX_aC = \angle BX_aM_a + \angle M_aX_aC = \angle B + \angle C = 180^\circ - \angle A.$$

Note also that X_a is on the same side of M_a as A. So X_a belongs to the circumcircle of BHC, where H is the orthocenter.

Because X_a is on the median, $\angle HX_aM_a = 90^\circ$. This can be proved in two ways: If we extend AM_a to meet the circumcircle of BHC at A', then ABC and $A'CB$ are symmetric with respect to M_a, which proves that CA' is parallel to AB and hence perpendicular to HC. By inscribed angles

$$\angle HX_aM_a = \angle HX_aA' = \angle HCA' = 90^\circ.$$

Or we can argue as follows. Denote by D, E, F the feet of the perpendiculars from vertices A, B, and C. Consider inversion $\mathcal{I}$ with the pole A and power $\rho = AH \cdot HD = AF \cdot FB = AE \cdot EC$. Then the circle passing through B, C, H, X_a becomes the nine-point circle of triangle ABC. It follows that $\mathcal{I}(X_a) = M_a$ and $AX_a \cdot AM_a = AH \cdot HD$, yielding that DHX_aM_a is cyclic. Thus $\angle HX_aG = 90^\circ$.

We conclude that X_a, X_b, X_c lie on a circle of diameter GH, and hence their circumcenter is on the Euler line GH.

§3 Solution to TSTST Problem 3

This problem was proposed by Alex Zhai.

For a prime q, we say a residue mod q is q-*abundant* if it appears in M infinitely often.

Suppose for sake of contradiction that there is a prime q in P but not in M. Let $S \subset M$ be the (finite) subset of primes whose residues are not q-abundant. Define $X = \prod_{p \in S} p$, and let a be the remainder of X modulo q.

Let $H \subset \mathbb{Z}/q$ denote the subset of residues that can be obtained by multiplying finitely many (not necessarily distinct) q-abundant residues. The definition of H ensures that for any $h \in H$, we can find a finite subset $U_h \subset M \setminus S$ such that $\prod_{p \in U_h} p \equiv h \pmod{q}$.

Now, consider the number $N = X \cdot \prod_{p \in U_h} p + 1$. Each prime factor of N must be in M. However, N is relatively prime to X, so it can only have prime factors whose residues are abundant. It follows that the residue of N modulo q is in H. Computing this, we find that

$$N \equiv X \cdot \prod_{p \in U_h} p + 1 \equiv a \cdot h + 1 \pmod{q},$$

so $ah + 1 \in H$.

This holds for each $h \in H$, so $h \mapsto ah + 1$ maps H to itself. Moreover, this map is a bijection, because

$$ah + 1 \equiv ah' + 1 \pmod q \implies h \equiv h' \pmod q.$$

However, clearly $1 \in H$ and

$$ah + 1 \equiv 1 \pmod q \implies h \equiv 0 \pmod q,$$

which is a contradiction.

§4 Solution to TSTST Problem 4

This problem was proposed by Alyazeed Basyoni.

We prove that the condition $x^4 + y^4 + z^4 + xyz = 4$ implies

$$\sqrt{2 - x} \ge \frac{y + z}{2}.$$

We first establish that $2 - x \ge 0$. Indeed, AM-GM gives that

$$\begin{aligned} 5 = x^4 + y^4 + (z^4 + 1) + xyz &= \frac{3x^4}{4} + \left(\frac{x^4}{4} + y^4\right) + (z^4 + 1) + xyz \\ &\ge \frac{3x^4}{4} + x^2y^2 + 2z^2 + xyz. \end{aligned}$$

We evidently have that $x^2y^2 + 2z^2 + xyz \ge 0$ because the quadratic form $a^2 + ab + 2b^2$ is positive definite, so $x^4 \le \frac{20}{3} \implies x \le 2$. Now, the desired statement is implied by its square, so it suffices to show that

$$2 - x \ge \left(\frac{y + z}{2}\right)^2$$

Assume for contradiction the reverse inequality holds. This rearranges to

$$4x + y^2 + 2yz + z^2 > 8.$$

By AM-GM, we have $x^4 + 3 \ge 4x$ and $\frac{y^4+1}{2} \ge y^2$ which yields that

$$x^4 + \frac{y^4 + z^4}{2} + 2yz + 4 > 8 \implies x^4 + \frac{y^4 + z^4}{2} + 2yz > 4.$$

Subtracting the given condition $x^4 + y^4 + z^4 + xyz = 4$ now gives

$$-\frac{y^4 + z^4}{2} + (2 - x)yz > 0 \implies (2 - x)yz > \frac{y^4 + z^4}{2}.$$

Since $2 - x$ and the right-hand side are positive, we have $yz \ge 0$. So, we have

$$\frac{y^4 + z^4}{2yz} < 2 - x < \left(\frac{y + z}{2}\right)^2 \implies 2y^4 + 2z^4 < yz(y + z)^2 = y^3z + 2y^2z^2 + yz^3.$$

This is clearly false by AM-GM, so we have a contradiction.

§5 Solution to TSTST Problem 5

This problem was proposed by Iurie Boreico.

Let a be a positive integer. Define a relation on the integers by saying that two numbers are *a-equal* iff their prime factorizations only differ by primes strictly less than a. (Thus for primes p greater than or equal to a, two a-equal numbers are divisible by the same power of p.) We will say that a set of integers $\{n_1, n_2, \dots, n_k\}$ is *a-distinct* if no two of them are a-equal. Finally, we say that $\{n_1, n_2, \dots, n_k\}$ are *a-compatible* if $\phi(n_1), \dots, \phi(n_k)$ are all a-equal.

Claim. If there exist k a-distinct, a-compatible integers $x_1, \dots, x_k$, then there exist k distinct integers $n_1, \dots, n_k$ such that $\phi(n_1), \dots, \phi(n_k)$ are all equal.

Proof. Suppose that $x_1, \dots, x_k$ are a-distinct and a-compatible. Let p be the first prime smaller than a. Then $x_1, \dots, x_k$ are p-distinct, but may not be p-compatible, because $\phi(x_1), \dots, \phi(x_k)$ may be divisible by different powers of p. However, if we multiply $x_1, \dots, x_k$ by appropriate powers of p, they will become p-compatible. Thus, continuing by downward induction on the primes less than a, we will arrive at integers $n_1, \dots, n_k$ which are 2-distinct and 2-compatible, which is the same as being distinct with $\phi(n_1), \dots, \phi(n_k)$ all equal.

Now set $a = 2^k$ (we can eventually take $k = 2015$). By Bertrand's postulate, there exist k primes $p_1, \dots, p_k$ between a and a^2. We will take $p_1, \dots, p_k$ to be the smallest primes larger than a. Consider one of these primes p_i. Then $\phi(p_i) = p_i - 1$, so that $a \le \phi(p_i) < a^2$, so that $\phi(p_i)$ is divisible by at most one other prime among the other $k-1$ primes, and all the other factors of $\phi(p_i)$ are less than a. Thus we can form a directed graph with an arrow from p_i to p_j iff p_j divides $\phi(p_i)$. Clearly this graph is a union of connected components, each of which is a tree.

For each component C_r containing c_r primes, we will find $c_r + 1$ a-distinct, a-compatible integers. Now let $p_1, \dots, p_{c_r}$ be the primes in C_r. If there is an arrow from p_i to p_j, define $\hat{p_i} = p_j$. If p_i has no arrow coming out of it, define $\hat{p_i} = 1$. Now consider $P_0 = \prod_{i=1}^{c_r} p_i$. Let $P_i = P_0 \cdot \frac{\hat{p_i}}{p_i}$. By construction, $P_0, P_1, \dots, P_{c_r}$ are all a-distinct: P_0 is divisible by all the primes in C_r, while each of the other P_i is missting a different prime. Moreover, $P_0, P_1, \dots, P_{c_r}$ are all a-compatible, because $\phi(p_i \cdot \hat{p_i})$ and $\phi(\hat{p_i}^2)$ are a-compatible.

Thus each component C_r gives rise to $c_r + 1$ a-distinct, a-compatible integers. Let us now call them $P_{r,0}, P_{r,1}, \dots, P_{r,c_r}$. Note that the prime factors of these integers are completely disjoint from component to component. Consider all the products consisting of one $P_{r,i}$ for each value of r. There are $\prod(c_r + 1) > \sum c_r = k$ of these products. Moreover, because $P_{r,i}$ is relatively prime to $P_{r',i'}$ whenever $r \ne r'$, these products are all a-compatible and a-distinct. Thus we have found at least k a-distinct and a-compatible integers, and we are done. □

There are many ways of finding 2015 a-compatible, a-distinct integers. For example,

$$5,\ 7,\ 13,\ 17,\ 19,\ 37,\ 73,\ 97,\ 109,\ 163,\ 193,\ 257$$

are a set of twelve 4-distinct, 4-compatible integers. Taking all 2^{12} combinations of products gives $2^{12} > 2015$ 4-distinct, 4-compatible integers. Similarly,

$$7,\ 11,\ 13,\ 17,\ 19,\ 31,\ 37,\ 41,\ 61,\ 73,\ 97,\ 101$$

are all 6-compatible and 6-distinct.

§6 Solution to TSTST Problem 6

This problem was proposed by Linus Hamilton.

Here we present a solution with 14 registers and 22 moves. Initially $X = n$ and all other variables are zero.

	X	Y	Go	S_X^0	S_X	S'_X	S_Y^0	S_Y	S'_Y	Cl	A	B	Die	Die'
Init	-1		1									1	1	1
Begin	1		-1	1							-1	1		
Sleep											1	-1		
StartX				-1	1						-1	1		
WorkX	-1				-1	1					-1	1		
WorkX'	-1	1			1	-1					-1	1		
DoneX					-1		1				-1	1		
WrX		-1		-1			-1	1			1	-1		
StartY							-1	1			-1	1		
WorkY		-1						-1	1		-1	1		
WorkY'	1	-1						1	-1		-1	1		
DoneY				1				-1			-1	1		
WrY	-1						-1				1	-1		
ClaimX	-1			-1						1	-1	1		
ClaimY		-1					-1			1	-1	1		
FakeX	-1									-1		-1		
FakeY		-1								-1		-1		
Win										-1	-1			
PunA												-2		
PunB											-1	-1		
Kill												-1	-2	1
Kill'												-1	1	-2

Now, the "game" is played as follows. First, note that Alice must play Init, and then Bob must play Sleep. From then on the mechanics are controlled by the *turn counters* A and B. To be precise, we say that the game is

- In the *main part* if $A + B = 1$, and no one has played Init a second time.
- In the *death part* otherwise.

Observe that in the main state, on Alice's turn we always have $(A, B) = (1, 0)$ and on Bob's turn we always have $(A, B) = (0, 1)$.

Claim. A player whose move leaves the game in the death part must lose.

Proof.
- Suppose the offending player is in a situation where $(A, B) = (0, 0)$. Then he/she must play Init. At this point, the opposing player can respond by playing Kill. Then the offending player must play Init again. The opposing player now responds with Kill'. This iteration continues until InX reaches a negative number and the offending player loses.
- Suppose Alice has $(A, B) = (1, 0)$ but plays Init again anyways. Then Bob responds with PunA to punish her; he then wins as in the first case.
- Suppose Bob has $(A, B) = (0, 1)$ but plays Init again anyways. Alice responds with PunB in the same way.

Situations with $A + B \geq 2$ cannot occur during main part, so this is all cases. □

Now we return to analysis of the main part. Observe the main part starts with Alice playing Init, Bob playing Sleep, and then Alice playing Begin (thus restoring the value of n in X), then Bob playing Sleep.

Thereafter we say the game is in *state* S for $S \in \{S_X^0, S_X, S_X', S_Y^0, S_Y, S_Y', \mathrm{Cl}\}$ if $S = 1$ and all other variables are zero. By construction, this is always the case. From then on the main part is divided into several phases:

- An X*-phase*: this begins with Alice at S_X^0, and ends when the game is in a state other than S_X and S_X'. (She can never return to S_X^0 during an X-phase.)
- A Y*-phase*: this begins with Alice at S_Y^0, and ends when the game is in a state other than S_Y and S_Y'. (She can never return to S_Y^0 during a Y-phase.)

Claim. Consider an X-phase in which $(X, Y) = (x, 0)$, $x > 1$. Then Alice can complete the phase without losing if and only if x is even; if so she begins a Y-phase with $(X, Y) = (0, x/2)$.

Proof. As $x > 1$, Alice cannot play ClaimX since Bob will respond with FakeX and win. Now by alternating between WorkX and WorkX', Alice can repeatedly deduct 2 from X and add 1 to Y, leading to $(x-2, y+1)$, $(x-4, y+2)$. (During this time, Bob can only play Sleep.) Eventually, she must stop this process by playing DoneX, which begins a Y-phase.

Now note that unless $X = 0$, Bob now has a winning move WrY ("wrong Y-phase"). Conversely he may only play Sleep if $X = 0$. □

We have an analogous claim for Y-phases. Thus if n is not a power of 2, we see that Alice eventually loses.

Now suppose $n = 2^k$ is a power of 2, then Alice reaches $(X, Y) = (0, 2^{k-1})$, $(2^{k-2}, 0)$, ... until either reaching $(1, 0)$ or $(0, 1)$. At this point she can play ClaimX or ClaimY, respectively; the game is now in state Cl. Bob cannot play either FakeX or FakeY, so he must play Sleep, and then Alice wins by playing Win. Thus Alice has a winning strategy when $n = 2^k$.