Team Selection Test for the 54th IMO
December 13, 2012

1. A social club has $2k + 1$ members, each of whom is fluent in the same $k$ languages. Any pair of members always talk to each other in only one language. Suppose that there were no three members such that they use only one language among them. Let $A$ be the number of three-member subsets such that the three distinct pairs among them use different languages. Find the maximum possible value of $A$.

2. Find all triples $(x, y, z)$ of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

3. Let $ABC$ be a scalene triangle with $\angle BCA = 90^\circ$, and let $D$ be the foot of the altitude from $C$. Let $X$ be a point in the interior of the segment $CD$. Let $K$ be the point on the segment $AX$ such that $BK = BC$. Similarly, let $L$ be the point on the segment $BX$ such that $AL = AC$. The circumcircle of triangle $DKL$ intersects segment $AB$ at a second point $T$ (other than $D$). Prove that $\angle ACT = \angle BCT$.

4. Let $f$ be a function from positive integers to positive integers, and let $f^m$ be $f$ applied $m$ times. Suppose that for every positive integer $n$ there exists a positive integer $k$ such that $f^{2k}(n) = n + k$, and let $k_n$ be the smallest such $k$. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.
1. Two incongruent triangles $ABC$ and $XYZ$ are called a pair of *pals* if they satisfy the following conditions:

(a) the two triangles have the same area;

(b) let $M$ and $W$ be the respective midpoints of sides $BC$ and $YZ$. The two sets of lengths \{\(AB, AM, AC\)\} and \{\(XY, XW, XZ\)\} are identical 3-element sets of pairwise relatively prime integers.

Determine if there are infinitely many pairs of triangles that are pals of each other.

2. Let $ABC$ be an acute triangle. Circle $\omega_1$, with diameter $AC$, intersects side $BC$ at $F$ (other than $C$). Circle $\omega_2$, with diameter $BC$, intersects side $AC$ at $E$ (other than $C$). Ray $AF$ intersects $\omega_2$ at $K$ and $M$ with $AK < AM$. Ray $BE$ intersects $\omega_1$ at $L$ and $N$ with $BL < BN$. Prove that lines $AB$, $ML$, $NK$ are concurrent.

3. In a table with $n$ rows and $2n$ columns where $n$ is a fixed positive integer, we write either zero or one into each cell so that each row has $n$ zeros and $n$ ones. For $1 \leq k \leq n$ and $1 \leq i \leq n$, we define $a_{k,i}$ so that the $i^{th}$ zero in the $k^{th}$ row is the $a_{k,i}^{th}$ column. Let $\mathcal{F}$ be the set of such tables with $a_{1,i} \geq a_{2,i} \geq \cdots \geq a_{n,i}$ for every $i$ with $1 \leq i \leq n$. We associate another $n \times 2n$ table $f(C)$ from $C \in \mathcal{F}$ as follows: for the $k^{th}$ row of $f(C)$, we write $n$ ones in the columns $a_{n,k} - k + 1, a_{n-1,k} - k + 2, \ldots, a_{1,k} - k + n$ (and we write zeros in the other cells in the row).

(a) Show that $f(C) \in \mathcal{F}$.

(b) Show that $f(f(f(f(f(C)))))) = C$ for any $C \in \mathcal{F}$.

4. Determine if there exists a (three-variable) polynomial $P(x, y, z)$ with integer coefficients satisfying the following property: a positive integer $n$ is *not* a perfect square if and only if there is a triple $(x, y, z)$ of positive integers such that $P(x, y, z) = n$. 