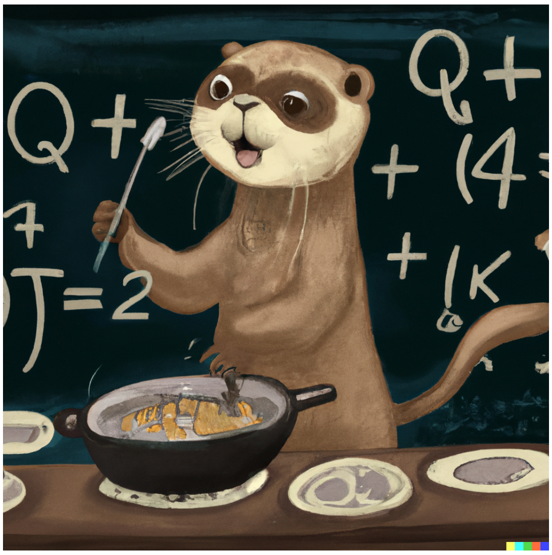
OTIS Mock AIME II 2025

https://web.evanchen.cc/mockaime.html

Evan Chen《陳誼廷》

20 December 2024



Instructions

1. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

2. No aids other than writing utensils and erasers, scratch paper, graph paper, ruler, compass, protractor are permitted. In particular, books, notes, calculators, cell phones, computers, abacuses, ChatGPT, magic crystal balls, etc., are all prohibited.

3. Problem statements follow ARML conventions, so for example $\lfloor x \rfloor$ is the floor of x. You may refer to these conventions during the exam.

4. You may submit your answers at https://forms.gle/RmTJ8WHtiuuB3Tzj9 up until the deadline of 15 January 2025 23:59 Pacific time. Shortly after that answers and solutions will be posted on Evan's website. Please avoid public discussion of problems before that date. (Private discussions with people who have finished or do not plan to submit is OK.)

Again, this is **Test II** of the OTIS Mock AIME. Good luck!

Problems

Problem II.1. Let ABC be a triangle with $\angle B = 60^\circ$ and $AB = 8$. Let D be the foot of the altitude from A to BC, and let M be the midpoint of CD. If $AM = BM$, compute AC^2.

Problem II.2. Let P denote the product of all positive integers n such that the least common multiple of $2n$ and n^2 is $62n - 336$. Compute the remainder when P is divided by 1000.

Problem II.3. Let x be the unique positive real number satisfying

$$2^x + 32^x = 8^x + 16^x.$$

Compute $8^{x+2} - 2^{x+6}$.

Problem II.4. Compute the number of integers less than 1000 which can be written in the form

$$x^{\lfloor x \rfloor} + \lfloor x \rfloor^x$$

for some positive real number x.

Problem II.5. Rosa the otter is stacking 53 blocks in a tower. For $n \ge 1$, after successfully placing the previous $n - 1$ blocks, the probability that placing the n^{th} block causes the whole tower to topple is $\frac{1}{54-n}$. Compute the expected number of blocks placed successfully before the block that causes the tower to topple.

Problem II.6. Compute the number of functions $f : \{1, 2, \dots, 15\} \to \{-1, 0, 1\}$ such that $f(ab) = f(a)f(b)$ holds whenever a and b are positive integers with $ab \le 15$.

Problem II.7. Vikram has a sheet of paper with all the numbers from 1 to 1000 written on it in a row. He then removes every multiple of 6 or 7. In doing so, the remaining numbers are split up into contiguous runs of consecutive numbers, such as $\{1, 2, 3, 4, 5\}$, $\{25, 26, 27\}$, or $\{13\}$. The average length of a run can be written as $\frac{p}{q}$ for relatively prime positive integers p and q. Compute $p + q$.

Problem II.8. Let ABC be an equilateral triangle with side length 600, and let P be a point on the circumcircle of ABC such that $AP = 630$ and $PB > PC$. Let M be the midpoint of BC. Point D is chosen on line BP such that $MD \perp AP$. Compute PD.

Problem II.9. Let N denote the number of polynomials $P(x)$ of degree 3 and with leading coefficient 1 such that

- Every coefficient of $P(x)$ is an integer with absolute value at most 10;
- There exist two distinct integers m and n such that $P(mi) = P(ni)$. (Here $i = \sqrt{-1}$.)

Compute the remainder when N is divided by 1000.

Problem II.10. Let S be the set of positive integers that are divisible by either 14 or 34 (or both), but not by any prime that doesn't divide 14 or 34. (For example, $14 \cdot 34 \in S$, but $14 \cdot 3 \cdot 4 \notin S$.) Let $d(s)$ denote the number of positive integers dividing s. Suppose that

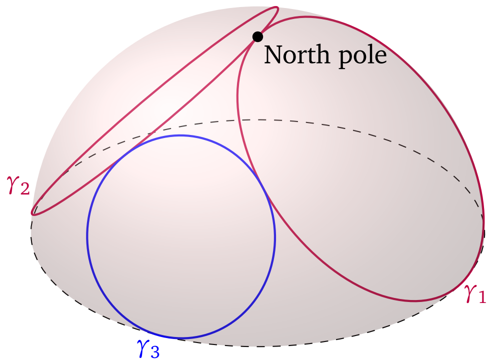
$$\sum_{s \in S} \frac{d(s)}{s} = \frac{pqr}{m}$$

for some primes p, q, r and a positive integer m. Compute $p + q + r$.

Problem II.11. At an informatics competition each student earns a score in $\{0, 1, \ldots, 100\}$ on each of six problems, and their total score is the sum of the six scores (out of 600). Given two students A and B, we write $A \succ B$ if there are at least five problems on which A scored strictly higher than B.

Compute the smallest integer c such that the following statement is true: for every integer $n \geq 2$, given students $A_1, \ldots, A_n$ satisfying $A_1 \succ A_2 \succ \cdots \succ A_n$, the total score of A_n is always at most c points more than the total score of A_1.

Problem II.12. Let γ_1, γ_2, and γ_3 be circles drawn on the surface of a hemisphere with radius 10. Each circle is tangent to the base of the hemisphere and pairwise tangent to one another. Additionally, γ_1 and γ_2 are congruent and tangent to each other at the north pole of the hemisphere, the point of the hemisphere farthest from the base. Compute the greatest integer less than the area of γ_3 (here, the area of γ_3 is taken with respect to the plane containing γ_3).



Problem II.13. Compute the largest positive integer m such that 2^m divides

$$\sum_{k=0}^{717} (-1)^k \binom{717}{k} (6 + 239k)^{717}.$$

Problem II.14. The incircle of ABC is tangent to BC at D. Let the internal bisectors of $\angle BAD$ and $\angle BDA$ meet at I_B and their external bisectors at E_B, and define I_C and E_C similarly. Suppose that $I_B I_C = 1$, $E_B E_C = 6$, and the area of quadrilateral $I_B I_C E_B E_C$ is 7. The area of triangle ABC can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Problem II.15. The Queen of Hearts has a special deck of 16 playing cards and a 4×4 square grid. Each card has one of four different ranks and one of four different suits, with each combination occurring exactly once. She wishes to place the cards in the grid, with one card in each cell, such that any cards in adjacent cells share either a rank or a suit. Compute the remainder when the number of ways to fill the grid is divided by 1000.