OTIS Mock AIME I 2025

https://web.evanchen.cc/mockaime.html

Evan Chen《陳誼廷》

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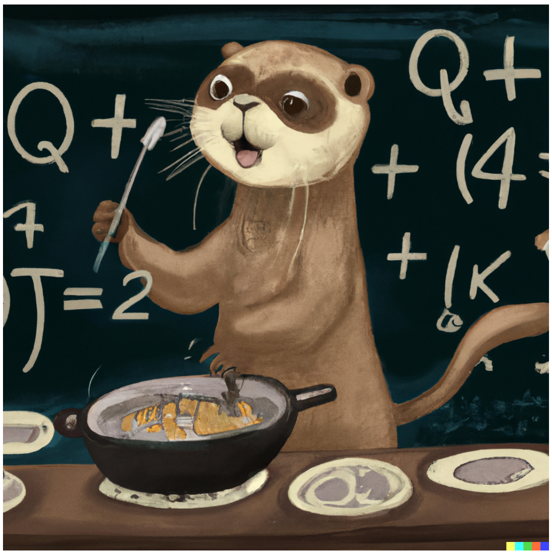

Instructions

1. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

2. No aids other than writing utensils and erasers, scratch paper, graph paper, ruler, compass, protractor are permitted. In particular, books, notes, calculators, cell phones, computers, abacuses, ChatGPT, magic crystal balls, etc., are all prohibited.

3. Problem statements follow ARML conventions, so for example $\lfloor x \rfloor$ is the floor of x. You may refer to these conventions during the exam.

4. You may submit your answers at https://forms.gle/nWowXxZWJvpzcVNw6 up until the deadline of 15 January 2025 23:59 Pacific time. Shortly after that answers and solutions will be posted on Evan's website. Please avoid public discussion of problems before that date. (Private discussions with people who have finished or do not plan to submit is OK.)

This is **Test I**, so it'll feel harder than an actual AIME. Stay determined and good luck!

Problems

Problem I.1. In a 3×3 grid, each cell is empty or contains a penguin. Two penguins are *angry* at each other if they occupy diagonally adjacent cells. Compute the number of ways to fill the grid so that none of the penguins are angry.

Problem I.2. Convex quadrilateral $ABCD$ has $AD = 72$, $\angle ABC = \angle ACD = 90^\circ$ and $\angle BAC = \angle CAD = 30^\circ$. Let M be the midpoint of AD and let N be the midpoint of BM. Compute CN^2.

Problem I.3. Compute the smallest integer $k > 1$ such that there are exactly 10 even integers $n \geq 2$ for which $k - n/2$ is divisible by n.

Problem I.4. In the Cartesian plane, let $A = (0, 10+12\sqrt{3})$, $B = (8, 10+12\sqrt{3})$, $G = (8,0)$ and $H = (0,0)$. Compute the number of ways to draw an equiangular dodecagon $\mathscr{P}$ in the Cartesian plane such that all side lengths of $\mathscr{P}$ are positive integers and line segments AB and GH are both sides of $\mathscr{P}$.

Problem I.5. Let x, y, and z be complex numbers satisfying

$$|x+z| = |y+z| = |x-y| = 4.$$

Compute $|x+2y+3z|^2$.

Problem I.6. There are 2025 green pencils on a table. Every minute, Elphaba removes two randomly chosen pencils on the table. Right after that, Glinda adds back one pink pencil. After 2023 minutes, the probability that at least one of the two pencils remaining on the table is green is $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute the remainder when $m+n$ is divided by 1000.

Problem I.7. Let ABC be a triangle with $AB = 5$, $BC = 13$, and $CA = 12$. Points D, E, and F are on segments BC, CA, and AB such that DEF is an isosceles right triangle with hypotenuse EF. Suppose that $BF = 3$. Then the length of CE can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Problem I.8. Compute the maximum possible value of $ab + bc + cd + de$ over all choices of positive integers a, b, c, d, e satisfying $a+b+c+d+e = 60$.

Problem I.9. Winston forgot the definition of a prime number. He instead defines a *New-prime* recursively as follows:

- 1 is not New-prime.
- A positive integer $n > 1$ is New-prime if and only if n cannot be expressed as the product of exactly two (not necessarily distinct) New-prime positive integers.

Compute the number of positive integers dividing 5005^4 which are New-primes.

Problem I.10. An 8×8 grid of unit squares is drawn; it thus has 144 unit edges. Let N be the number of ways to color each of the 144 unit edges one of six colors (red, orange, yellow, green, blue, or purple) such that every unit square is surrounded by exactly 3 different colors. Then N can be written as a prime factorization $p_1^{e_1} \dots p_k^{e_k}$ where $p_1 < \dots < p_k$ are primes and e_i are positive integers. Compute $e_1 + \dots + e_k$.

Problem I.11. Let ABC be an acute non-equilateral triangle with $\angle BAC = 60^\circ$. The Euler line of triangle ABC intersects side BC at point X such that B lies between X and C. Given that $XA = 49$ and $XB = 23$, compute XC.

(The *Euler line* of a non-equilateral triangle refers to the line through its circumcenter, centroid, and orthocenter.)

Problem I.12. There exists a unique tuple of positive real numbers (a, b, c, d) satisfying

$$\begin{aligned}(49+ab)(a+b) &= 81a+25b\\ (81+bc)(b+c) &= 121b+49c\\ (121+cd)(c+d) &= 169c+81d\\ a+b+c+d &= 12.\end{aligned}$$

Given that $d = m - \sqrt{n}$ for positive integers m and n, compute $m+n$.

Problem I.13. Let ABC be an acute triangle. Suppose the distances from its circumcenter, incenter, and orthocenter to side BC are 8, 6, and 4, respectively. Compute BC^2.

Problem I.14. There is a unique triplet of integers (a, b, c) such that $0 < a < b < c < 1000$ and

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{315}.$$

Compute a.

Problem I.15. Alice has a deck of 2000 cards, numbered 1 through 2000. Alice chooses an integer $1 \le n < 1000$ and deals Cheshire a random subset of $2n-1$ of the cards without repetition. Cheshire wins if the cards dealt contains any n consecutively numbered cards. Compute the value of n Alice should choose to minimize Cheshire's chances of winning.