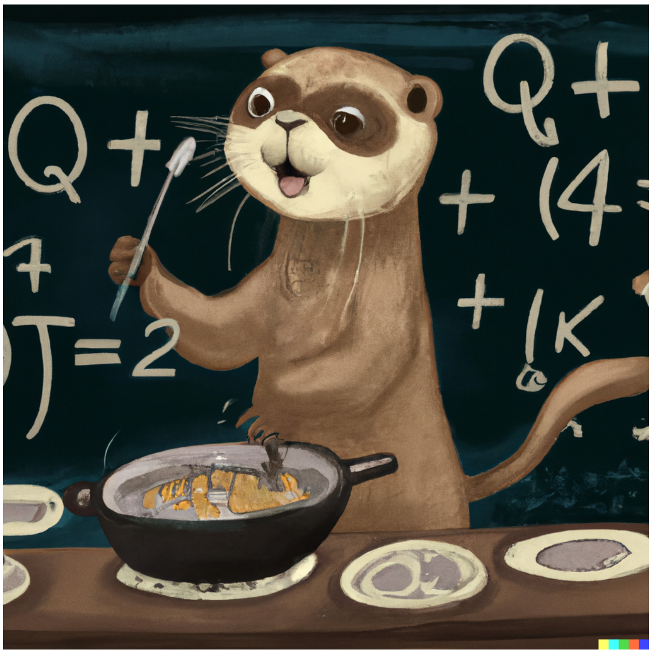
# OTIS Mock AIME 2024

https://web.evanchen.cc/mockaime.html

Evan Chen《陳誼廷》

22 December 2023




## Instructions

1. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

2. No aids other than writing utensils and erasers, scratch paper, graph paper, ruler, compass, protractor are permitted. In particular, books, notes, calculators, cell phones, computers, abacuses, ChatGPT, magic crystal balls, etc., are all prohibited.

3. Problem statements follow ARML conventions, so for example $\lfloor x \rfloor$ is the floor of $x$. You may refer to these conventions during the exam.

4. You may submit your answers at https://forms.gle/tdsfHKYnSzKA53TZA up until the deadline of 15 January 2024 23:59 Pacific time. Shortly after that answers and solutions will be posted on Evan's website. Please avoid public discussion of problems before that date. (Private discussions with people who have finished or do not plan to submit is OK.)

## Problems

**Problem 1.** Compute the number of real numbers $x$ such that $0 < x \le 100$ and

$$x^2 = \lfloor x \rfloor \cdot \lceil x \rceil.$$

**Problem 2.** Evan throws a dart at regular hexagon $BOATIS$, which lands at a uniformly random point inside the hexagon. The probability the dart lands in the interior of exactly one of the quadrilaterals $BOAT$ and $OTIS$ can be expressed as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m+n$.

**Problem 3.** Perry the Panda is eating some bamboo over a five-day period from Monday to Friday (inclusive). On Monday, he eats 14 pieces of bamboo. Each following day, Perry eats either one less than three times the previous day or one more than the previous day, with equal probability. Compute the expected number of pieces of bamboo Perry has eaten throughout the week after the end of Friday.

**Problem 4.** Let $N$ denote the number of 7-tuples of positive integers $(a_1, \dots, a_7)$ such that for each $i = 1, \dots, 7$, we have

$$1 \le a_i \le 7 \quad \text{and} \quad a_{a_i} = a_i.$$

Compute the remainder when $N$ is divided by 1000.

**Problem 5.** Convex pentagon $ABCDE$ is inscribed in circle $\omega$ such that $\frac{AC}{DE} = \frac{2}{3}$, $AE = CD$, and $AB = BC$. Suppose the distance from $B$ to line $AC$ is 144 and the distance from $B$ to line $DE$ is 864. Compute the radius of $\omega$.

**Problem 6.** For each real number $k > 0$, let $S(k)$ denote the set of real numbers $x$ satisfying

$$\lfloor x \rfloor \cdot (x - \lfloor x \rfloor) = kx.$$

The set of positive real numbers $k$ such that $S(k)$ has exactly 24 elements is a half-open interval of length $\ell$. Compute $1/\ell$.

**Problem 7.** Compute the number of 9-tuples $(a_0, a_1, \dots, a_8)$ of integers such that $a_i \in \{-1, 0, 1\}$ for $i = 0, 1, \dots, 8$ and the polynomial

$$a_8 x^8 + a_7 x^7 + \dots + a_1 x + a_0$$

is divisible by $x^2 + x + 1$.

**Problem 8.** Let $n \ge k \ge 1$ be integers such that the binomial coefficient $\binom{n}{k}$ is a multiple of 1000. Compute the smallest possible value of $n + k$.

**Problem 9.** Let $\omega$ be a circle with center $O$ and radius 12. Points $A$, $B$, and $C$ are chosen uniformly at random on the circumference of $\omega$. Let $H$ denote the orthocenter of $\triangle ABC$. Compute the expected value of $OH^2$.

**Problem 10.** Compute the number of integers $b \in \{1, 2, \dots, 1000\}$ for which there exists positive integers $a$ and $c$ satisfying

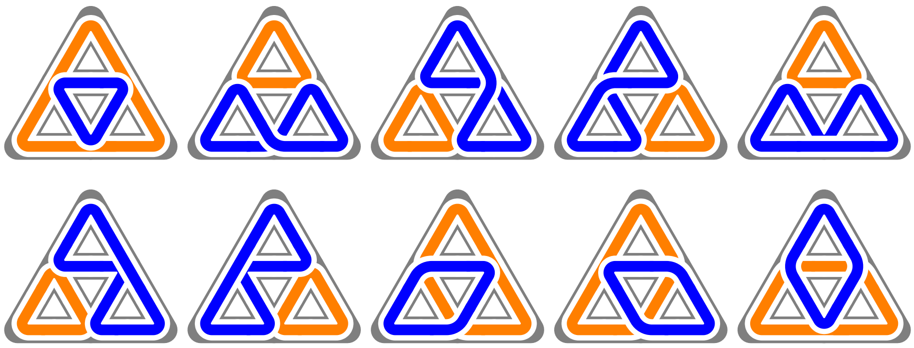
$$\gcd(a, b) + \operatorname{lcm}(b, c) = \operatorname{lcm}(c, a)^3.$$

**Problem 11.** Compute the number of ordered triples of positive integers $(a, b, n)$ satisfying $\max(a, b) \le \min(\sqrt{n}, 60)$ and

$$\operatorname{Arcsin}\left(\frac{a}{\sqrt{n}}\right) + \operatorname{Arcsin}\left(\frac{b}{\sqrt{n}}\right) = \frac{2\pi}{3}.$$

**Problem 12.** Let $\mathscr{G}_n$ denote a triangular grid of side length $n$ consisting of $\frac{(n+1)(n+2)}{2}$ pegs. Charles the Otter wishes to place some rubber bands along the pegs of $\mathscr{G}_n$ such that every edge of the grid is covered by exactly one rubber band (and no rubber band traverses an edge twice). He considers two placements to be different if the sets of edges covered by the rubber bands are different or if any rubber band traverses its edges in a different order. The ordering of which bands are over and under does not matter.

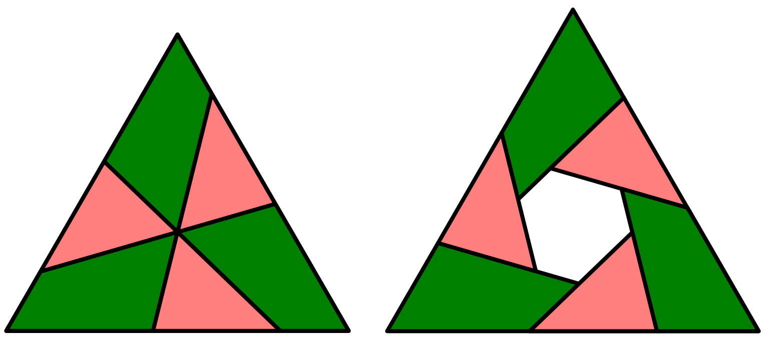
For example, Charles finds there are exactly 10 different ways to cover $\mathscr{G}_2$ using exactly two rubber bands; the full list is shown below, with one rubber band in orange and the other in blue.



Let $N$ denote the total number of ways to cover $\mathscr{G}_4$ with *any number* of rubber bands. Compute the remainder when $N$ is divided by 1000.

**Problem 13.** Let $S$ denote the sum of all integers $n$ such that $1 \le n \le 2024$ and exactly one of $n^{22}-1$ and $n^{40}-1$ is divisible by 2024. Compute the remainder when $S$ is divided by 1000.

**Problem 14.** Ritwin the Otter has a cardboard equilateral triangle. He cuts the triangle with three congruent line segments of length $x$ spaced at $120^\circ$ angles through the center, obtaining six pieces: three congruent triangles and three congruent quadrilaterals. He then flips all three triangles over, then rearranges all six pieces to form another equilateral triangle with an equiangular hexagonal hole inside it, as shown below.



Given that the side lengths of the hole are 3, 2, 3, 2, 3, 2, in that order, the value of $x$ can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m+n$.

**Problem 15.** A parabola in the Cartesian plane is tangent to the $x$-axis at $(1,0)$ and to the $y$-axis at $(0,3)$. The sum of the coordinates of the vertex of the parabola can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Compute $m+n$.

# Acknowledgments

## Software and infrastructure

Howard Halim.

## Editors

Amogh Akella, Arjun Suresh, Evan Chen, Joshua Liu, Kenny Tran, Royce Yao, Vincent Pirozzo.

## Problem proposers

Aidan Bai, Albert Cao, Alex Wang, Alex Yan, Amogh Akella, Anay Aggarwal, Arjun Suresh, Ashvin Sinha, Atticus Stewart, Ethan Lee, Ethan Wang, Haofang Zhu, Jiahe Liu, Jonathan He, Joshua Liu, Karn Chutinan, Kenny Tran, Kevin Liu, Leo Yu, Neil Kolekar, Oron Wang, Patrick Lu, Raymond Zhu, Royce Yao, Shining Sun, Skyler Mao, Tanishq Pauskar, Tarun Rapaka, Vikram Sarkar, Vincent Pirozzo, Wilbert Chu.

## Elite testsolvers

Anay Aggarwal, Atticus Stewart, Benjamin Jeter, Calvin Wang, Danielle Wang, Ethan Lee, Jiahe Liu, Karn Chutinan, Krishna Pothapragada, Liran Zhou, Pavan Jayaraman, Ritwin Narra, Rohan Garg, Selena Ge, Shaheem Samsudeen, Shining Sun, Soham Samanta, Tanishq Pauskar, Vikram Sarkar, Wilbert Chu.

## Hardcore testsolvers

Aarya Garimella, Aidan Bai, Alan Cheng, Aleksij Tasić, Alex Sun, Alex Yan, Alexander Wang, Amogh Akella, Anthony Zou, Ashvin Sinha, Benjamin Jump, Bhavya Tiwari, Ervin Joshua Bautista, Jackson Dryg, Jason Lee, Jonathan He, Jordan Lefkowitz, Joshua Liu, Karthik Vedula, Kevin Liu, Krishiv Khandelwal, Kyle Wu, Leo Yu, Likhith Malipati, Marin Hristov, Max May, Neil Kolekar, Praneel Samal, Raymond Zhu, Royce Yao, Ryan Wang, Sohil Rathi, Sophie Li, Tarun Rapaka, Vincent Pirozzo, Zongshu Wu.

## Testsolvers

Aarush Khare, Aaryan Vaishya, Abhinav Srinivas, Aditya Pahuja, Alan Yao, Alansha Jiang, Albert Cao, Alex Gu, Amol Rama, Andrew Lin, Anshul Mantri, Arjun Suresh, Arnav Adepu, Arul Kolla, Atharv Naphade, Ayush Bansal, Bole Ying, Daniel Chen, Elijah Liu, Ethan Wang, Evan Zhang, Greta Qu, Hansen Shieh, Haofang Zhu, Harry Kim, Jason Mao, Jerry Zhang, Kelin Zhu, Kenny Tran, Lincoln Liu, Lucas Qi, Luv Udeshi, Manu Parameshwaran, Oron Wang, Owen Zhang, Patrick Lu, Perry Dai, Rohan Shivakumar, Rohith Thomas, Ryan Chan, Shrey Sharma, Skyler Mao, Sounak Bagchi, Tawfiq Hod, Thomas Yao.