

# JMO 2016 Solution Notes

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February 13, 2020

This is an compilation of solutions for the 2016 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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## §0 Problems

1. The isosceles triangle  $\triangle ABC$ , with  $AB = AC$ , is inscribed in the circle  $\omega$ . Let  $P$  be a variable point on the arc  $BC$  that does not contain  $A$ , and let  $I_B$  and  $I_C$  denote the incenters of triangles  $\triangle ABP$  and  $\triangle ACP$ , respectively. Prove that as  $P$  varies, the circumcircle of triangle  $\triangle PI_B I_C$  passes through a fixed point.
2. Prove that there exists a positive integer  $n < 10^6$  such that  $5^n$  has six consecutive zeros in its decimal representation.
3. Let  $X_1, X_2, \dots, X_{100}$  be a sequence of mutually distinct nonempty subsets of a set  $S$ . Any two sets  $X_i$  and  $X_{i+1}$  are disjoint and their union is not the whole set  $S$ , that is,  $X_i \cap X_{i+1} = \emptyset$  and  $X_i \cup X_{i+1} \neq S$ , for all  $i \in \{1, \dots, 99\}$ . Find the smallest possible number of elements in  $S$ .
4. Find, with proof, the least integer  $N$  such that if any 2016 elements are removed from the set  $\{1, 2, \dots, N\}$ , one can still find 2016 distinct numbers among the remaining elements with sum  $N$ .
5. Let  $\triangle ABC$  be an acute triangle, with  $O$  as its circumcenter. Point  $H$  is the foot of the perpendicular from  $A$  to line  $BC$ , and points  $P$  and  $Q$  are the feet of the perpendiculars from  $H$  to the lines  $AB$  and  $AC$ , respectively.

Given that

$$AH^2 = 2AO^2,$$

prove that the points  $O$ ,  $P$ , and  $Q$  are collinear.

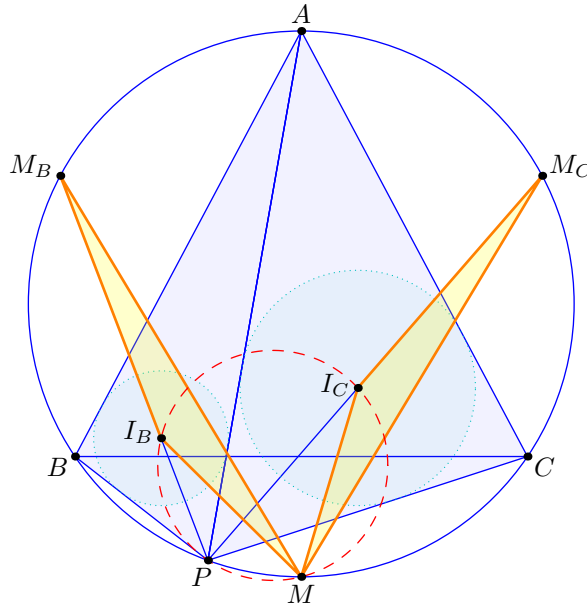
6. Find all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that for all real numbers  $x$  and  $y$ ,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

**§1 JMO 2016/1, proposed by Zuming Feng**

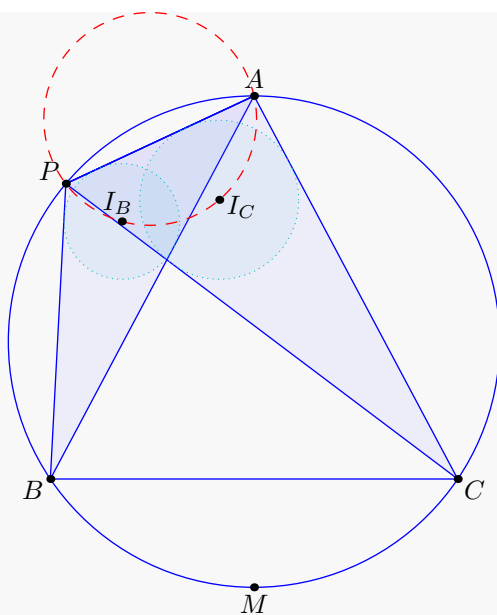
The isosceles triangle  $\triangle ABC$ , with  $AB = AC$ , is inscribed in the circle  $\omega$ . Let  $P$  be a variable point on the arc  $BC$  that does not contain  $A$ , and let  $I_B$  and  $I_C$  denote the incenters of triangles  $\triangle ABP$  and  $\triangle ACP$ , respectively. Prove that as  $P$  varies, the circumcircle of triangle  $\triangle PI_BI_C$  passes through a fixed point.

Let  $M$  be the midpoint of arc  $BC$  not containing  $A$ . We claim  $M$  is the desired fixed point.



Since  $\angle MPA = 90^\circ$  and ray  $PA$  bisects  $\angle I_BPI_C$ , it suffices to show that  $MI_B = MI_C$ . Let  $M_B, M_C$  be the second intersections of  $PI_B$  and  $PI_C$  with circumcircle. Now  $\widehat{M_BI_B} = \widehat{M_BB} = \widehat{M_CC} = \widehat{M_CI_C}$ , and moreover  $MM_B = MM_C$ , and  $\angle I_BM_BM = \frac{1}{2}\widehat{PM} = \angle I_CM_CM$ , so triangles  $\triangle I_BM_BM \cong \triangle I_CM_CM$ .

**Remark 1.1.** Complex in the obvious way DOES NOT WORK. See figure below.



Fun story, I pointed this out to Zuming during grading; I was the only one that realized the subtlety. We gave such solutions a 5.

## §2 JMO 2016/2, proposed by Evan Chen

Prove that there exists a positive integer  $n < 10^6$  such that  $5^n$  has six consecutive zeros in its decimal representation.

We will prove that  $n = 20 + 2^{19} = 524308$  fits the bill.

First, we claim that

$$5^n \equiv 5^{20} \pmod{5^{20}} \quad \text{and} \quad 5^n \equiv 5^{20} \pmod{2^{20}}.$$

Indeed, the first equality holds since both sides are  $0 \pmod{5^{20}}$ , and the second by  $\varphi(2^{20}) = 2^{19}$  and Euler's theorem. Hence

$$5^n \equiv 5^{20} \pmod{10^{20}}.$$

In other words, the last 20 digits of  $5^n$  will match the decimal representation of  $5^{20}$ , with leading zeros. However, we have

$$5^{20} = \frac{1}{2^{20}} \cdot 10^{20} < \frac{1}{1000^2} \cdot 10^{20} = 10^{-6} \cdot 10^{20}$$

and hence those first six of those 20 digits will all be zero. This completes the proof! (To be concrete, it turns out that  $5^{20} = 95367431640625$  and so the last 20 digits of  $5^n$  will be 00000095367431640625.)

**Remark.** Many of the first posts in the JMO 2016 discussion thread (see <https://aops.com/community/c5h1230514>) claimed that the problem was “super easy”. In fact, the problem was solved by only about 10% of contestants.

**Authorship comments** This problem was inspired by the observation  $5^8 \equiv 5^4 \pmod{10^4}$ , i.e. that  $5^8$  ended with 0625.

I noticed this one day back in November, when I was lying on my bed after a long afternoon and was mindlessly computing powers of 5 in my head because I was too tired to do much else. When I reached  $5^8$  I noticed for the first time that the ending 0625 was actually induced by  $5^4$ . (Given how much MathCounts I did, I really should have known this earlier!)

Thinking about this for a few more seconds, I realized one could obtain arbitrarily long strings of 0's by using a similar trick modulo larger powers of 10. This surprised me, because I would have thought that if this was true, then I would have learned about it back in my contest days. However, I could not find any references, and I thought the result was quite nice, so I submitted it as a proposal for the JMO, where I thought it might be appreciated.

The joke about six consecutive zeros is due to Zuming Feng.

### §3 JMO 2016/3, proposed by Iurie Boreico

Let  $X_1, X_2, \dots, X_{100}$  be a sequence of mutually distinct nonempty subsets of a set  $S$ . Any two sets  $X_i$  and  $X_{i+1}$  are disjoint and their union is not the whole set  $S$ , that is,  $X_i \cap X_{i+1} = \emptyset$  and  $X_i \cup X_{i+1} \neq S$ , for all  $i \in \{1, \dots, 99\}$ . Find the smallest possible number of elements in  $S$ .

Solution with Danielle Wang: the answer is that  $|S| \geq 8$ .

**Proof of sufficiency** Since we must have  $2^{|S|} \geq 100$ , we must have  $|S| \geq 7$ .

To see that  $|S| = 8$  is the minimum possible size, consider a chain on the set  $S = \{1, 2, \dots, 7\}$  satisfying  $X_i \cap X_{i+1} = \emptyset$  and  $X_i \cup X_{i+1} \neq S$ . Because of these requirements any subset of size 4 or more can only be neighbored by sets of size 2 or less, of which there are  $\binom{7}{1} + \binom{7}{2} = 28$  available. Thus, the chain can contain no more than 29 sets of size 4 or more and no more than 28 sets of size 2 or less. Finally, since there are only  $\binom{7}{3} = 35$  sets of size 3 available, the total number of sets in such a chain can be at most  $29 + 28 + 35 = 92 < 100$ .

**Construction** We will provide an inductive construction for a chain of subsets  $X_1, X_2, \dots, X_{2^{n-1}+1}$  of  $S = \{1, \dots, n\}$  satisfying  $X_i \cap X_{i+1} = \emptyset$  and  $X_i \cup X_{i+1} \neq S$  for each  $n \geq 4$ .

For  $S = \{1, 2, 3, 4\}$ , the following chain of length  $2^3 + 1 = 9$  will work:

$$34 \quad 1 \quad 23 \quad 4 \quad 12 \quad 3 \quad 14 \quad 2 \quad 13 \quad .$$

Now, given a chain of subsets of  $\{1, 2, \dots, n\}$  the following procedure produces a chain of subsets of  $\{1, 2, \dots, n+1\}$ :

1. take the original chain, delete any element, and make two copies of this chain, which now has even length;
2. glue the two copies together, joined by  $\emptyset$  in between; and then
3. insert the element  $n+1$  into the sets in alternating positions of the chain starting with the first.

For example, the first iteration of this construction gives:

$$\begin{array}{cccccccc} 345 & 1 & 235 & 4 & 125 & 3 & 145 & 2 & 5 \\ 34 & 15 & 23 & 45 & 12 & 35 & 14 & 25 & \end{array}$$

It can be easily checked that if the original chain satisfies the requirements, then so does the new chain, and if the original chain has length  $2^{n-1} + 1$ , then the new chain has length  $2^n + 1$ , as desired. This construction yields a chain of length 129 when  $S = \{1, 2, \dots, 8\}$ .

**Remark.** Here is the construction for  $n = 8$  in its full glory.

345678	1	235678	4	125678	3	145678	2	5678
34	15678	23	45678	12	35678	14	678	
345	1678	235	4678	125	3678	145	2678	5
34678	15	23678	45	12678	35	78		
3456	178	2356	478	1256	378	1456	278	56
3478	156	2378	456	1278	356	1478	6	
34578	16	23578	46	12578	36	14578	26	578
346	1578	236	4578	126	8			
34567	18	23567	48	12567	38	14567	28	567
348	1567	238	4567	128	3567	148	67	
3458	167	2358	467	1258	367	1458	267	58
3467	158	2367	458	1267	358	7		
34568	17	23568	47	12568	37	14568	27	568
347	1568	237	4568	127	3568	147	68	
3457	168	2357	468	1257	368	1457	268	57
3468	157	2368	457	1268				

## §4 JMO 2016/4, proposed by Gregory Galperin

Find, with proof, the least integer  $N$  such that if any 2016 elements are removed from the set  $\{1, 2, \dots, N\}$ , one can still find 2016 distinct numbers among the remaining elements with sum  $N$ .

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The answer is

$$N = 2017 + 2018 + \dots + 4032 = 1008 \cdot 6049 = 6097392.$$

To see that  $N$  must be at least this large, consider the situation when  $1, 2, \dots, 2016$  are removed. Among the remaining elements, any sum of 2016 elements is certainly at least  $2017 + 2018 + \dots + 4032$ .

Now we show this value of  $N$  works. Consider the 3024 pairs of numbers  $(1, 6048), (2, 6047), \dots, (3024, 3025)$ . Regardless of which 2016 elements of  $\{1, 2, \dots, N\}$  are deleted, at least  $3024 - 2016 = 1008$  of these pairs have both elements remaining. Since each pair has sum 6049, we can take these pairs to be the desired numbers.



### §5 JMO 2016/5, proposed by Zuming Feng and Jacek Fabrykowski

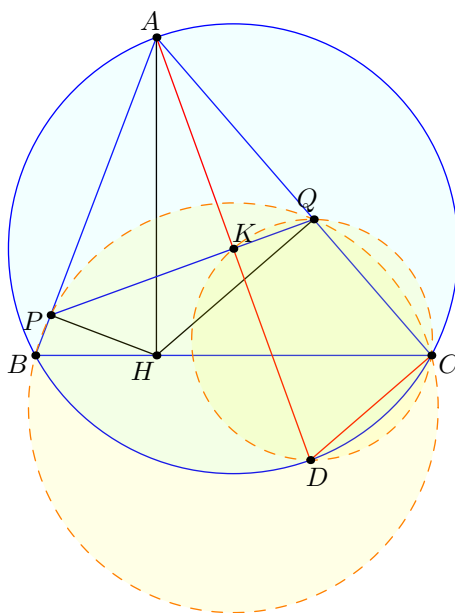
Let  $\triangle ABC$  be an acute triangle, with  $O$  as its circumcenter. Point  $H$  is the foot of the perpendicular from  $A$  to line  $BC$ , and points  $P$  and  $Q$  are the feet of the perpendiculars from  $H$  to the lines  $AB$  and  $AC$ , respectively.

Given that

$$AH^2 = 2AO^2,$$

prove that the points  $O$ ,  $P$ , and  $Q$  are collinear.

First, since  $AP \cdot AB = AH^2 = AQ \cdot AC$ , it follows that  $PQCB$  is cyclic. Consequently, we have  $AO \perp PQ$ .



Let  $K$  be the foot of  $A$  onto  $PQ$ , and let  $D$  be the point diametrically opposite  $A$ . Thus  $A, K, O, D$  are collinear.

Since quadrilateral  $KQCD$  is cyclic ( $\angle QKD = \angle QCD = 90^\circ$ ), we have

$$AK \cdot AD = AQ \cdot AC = AH^2 \implies AK = \frac{AH^2}{AD} = \frac{AH^2}{2AO} = AO$$

so  $K = O$ .

## §6 JMO 2016/6, proposed by Titu Andreescu

Find all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that for all real numbers  $x$  and  $y$ ,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

We claim that the only two functions satisfying the requirements are  $f(x) \equiv 0$  and  $f(x) \equiv x^2$ . These work.

First, taking  $x = y = 0$  in the given yields  $f(0) = 0$ , and then taking  $x = 0$  gives  $f(y)f(-y) = f(y)^2$ . So also  $f(-y)^2 = f(y)f(-y)$ , from which we conclude  $f$  is even. Then taking  $x = -y$  gives

$$\forall x \in \mathbb{R} : \quad f(x) = x^2 \quad \text{or} \quad f(4x) = 0 \quad (\star)$$

for all  $x$ .

Now we claim

$$\text{Claim — } f(z) = 0 \iff f(2z) = 0 \quad (\spadesuit).$$

*Proof.* Let  $(x, y) = (3t, t)$  in the given to get

$$(f(t) + 3t^2) f(8t) = f(4t)^2.$$

Now if  $f(4t) \neq 0$  (in particular,  $t \neq 0$ ), then  $f(8t) \neq 0$ . Thus we have  $(\spadesuit)$  in the forwards direction.

Then  $f(4t) \neq 0 \xrightarrow{(\star)} f(t) = t^2 \neq 0 \xrightarrow{(\spadesuit)} f(2t) \neq 0$  implies the reverse direction, the last step being the forward direction  $(\spadesuit)$ .  $\square$

By putting together  $(\star)$  and  $(\spadesuit)$  we finally get

$$\forall x \in \mathbb{R} : \quad f(x) = x^2 \quad \text{or} \quad f(x) = 0 \quad (\heartsuit)$$

We are now ready to approach the main problem. Assume there's an  $a \neq 0$  for which  $f(a) = 0$ ; we show that  $f \equiv 0$ .

Let  $b \in \mathbb{R}$  be given. Since  $f$  is even, we can assume without loss of generality that  $a, b > 0$ . Also, note that  $f(x) \geq 0$  for all  $x$  by  $(\heartsuit)$ . By using  $(\spadesuit)$  we can generate  $c > b$  such that  $f(c) = 0$  by taking  $c = 2^n a$  for a large enough integer  $n$ . Now, select  $x, y > 0$  such that  $x - 3y = b$  and  $x + y = c$ . That is,

$$(x, y) = \left( \frac{3c + b}{4}, \frac{c - b}{4} \right).$$

Substitution into the original equation gives

$$0 = (f(x) + xy) f(b) + (f(y) + xy) f(3x - y) = (f(x) + f(y) + 2xy) f(b)$$

Since  $f(x) + f(y) + 2xy > 0$ , it follows that  $f(b) = 0$ , as desired.