JMO 2013 Solution Notes

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This is a compilation of solutions for the 2013 JMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Are there integers a and b such that a^5b+3 and ab^5+3 are both perfect cubes of integers?

2. Each cell of an $m \times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be *adjacent* if their cells share a common side. The filling is called a *garden* if it satisfies the following two conditions:

   (i) The difference between any two adjacent numbers is either 0 or 1.

   (ii) If a number is less than or equal to all of its adjacent numbers, then it is equal to 0.

   Determine the number of distinct gardens in terms of m and n.

3. In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

4. Let $f(n)$ be the number of ways to write n as a sum of powers of 2, where we keep track of the order of the summation. For example, $f(4) = 6$ because 4 can be written as 4, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Find the smallest n greater than 2013 for which $f(n)$ is odd.

5. Quadrilateral $XABY$ is inscribed in the semicircle ω with diameter $\overline{XY}$. Segments AY and BX meet at P. Point Z is the foot of the perpendicular from P to line $\overline{XY}$. Point C lies on ω such that line XC is perpendicular to line AZ. Let Q be the intersection of segments AY and XC. Prove that

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$

6. Find all real numbers $x, y, z \ge 1$ satisfying

$$\min\left(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\right) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

§1 Solutions to Day 1

§1.1 JMO 2013/1, proposed by Titu Andreescu

Available online at https://aops.com/community/p3041819.

Problem statement

Are there integers a and b such that a^5b+3 and ab^5+3 are both perfect cubes of integers?

No, there do not exist such a and b.

We prove this in two cases.

- Assume $3 \mid ab$. WLOG we have $3 \mid a$, but then $a^5b+3 \equiv 3 \pmod 9$, contradiction.
- Assume $3 \nmid ab$. Then a^5b+3 is a cube not divisible by 3, so it is ± 1 mod 9, and we conclude

$$a^5b \in \{5,7\} \pmod 9.$$

Analogously

$$ab^5 \in \{5,7\} \pmod 9.$$

We claim however these two equations cannot hold simultaneously. Indeed $(ab)^6 \equiv 1 \pmod 9$ by Euler's theorem, despite $5 \cdot 5 \equiv 7$, $5 \cdot 7 \equiv 8$, $7 \cdot 7 \equiv 4$ mod 9.

§1.2 JMO 2013/2, proposed by Sungyoon Kim

Available online at https://aops.com/community/p3041818.

Problem statement

Each cell of an $m \times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be *adjacent* if their cells share a common side. The filling is called a *garden* if it satisfies the following two conditions:

(i) The difference between any two adjacent numbers is either 0 or 1.

(ii) If a number is less than or equal to all of its adjacent numbers, then it is equal to 0.

Determine the number of distinct gardens in terms of m and n.

The numerical answer is $2^{mn} - 1$. But we claim much more, by giving an explicit description of all gardens:

> Let S be any nonempty subset of the mn cells. Suppose we fill each cell θ with the minimum (taxicab) distance from θ to some cell in S (in particular, we write 0 if $\theta \in S$). Then
>
> - This gives a garden, and
> - All gardens are of this form.

Since there are $2^{mn} - 1$ such nonempty subsets S, this would finish the problem. An example of a garden with $|S| = 3$ is shown below.

$$\begin{bmatrix} 2 & 1 & 2 & 1 & \mathbf{0} & 1 \\ 1 & \mathbf{0} & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 3 & 2 & 3 \\ \mathbf{0} & 1 & 2 & 3 & 3 & 4 \end{bmatrix}$$

It is actually fairly easy to see that this procedure always gives a garden; so we focus our attention on showing that every garden is of this form.

Given a garden, note first that it has at least one cell with a zero in it — by considering the minimum number across the entire garden. Now let S be the (thus nonempty) set of cells with a zero written in them. We contend that this works, i.e. the following sentence holds:

Claim — If a cell θ is labeled d, then the minimum distance from that cell to a cell in S is d.

Proof. The proof is by induction on d, with $d = 0$ being by definition. Now, consider any cell θ labeled $d \geq 1$. Every neighbor of θ has label at least $d - 1$, so any path will necessarily take $d - 1$ steps after leaving θ. Conversely, there is *some* $d - 1$ adjacent to θ by (ii). Stepping on this cell and using the minimal path (by induction hypothesis) gives us a path to a cell in S with length *exactly* d. So the shortest path does indeed have distance d, as desired. $\square$

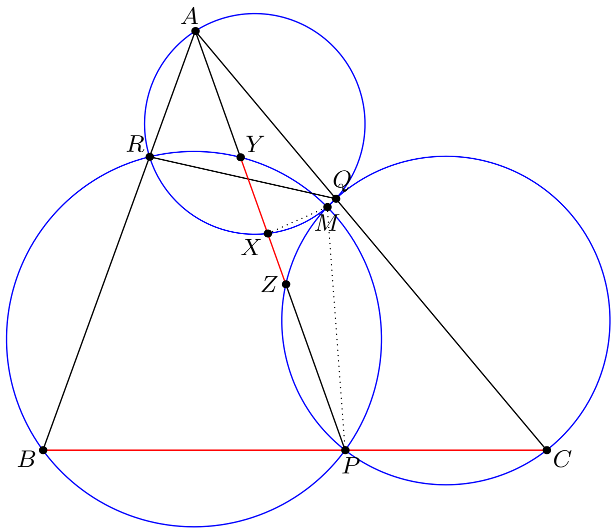
§1.3 JMO 2013/3, proposed by Zuming Feng

Available online at https://aops.com/community/p3041822.

Problem statement

In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

Let M be the concurrence point of ω_A, ω_B, ω_C (by Miquel's theorem).



Then M is the center of a spiral similarity sending $\overline{YZ}$ to $\overline{BC}$. So it suffices to show that this spiral similarity also sends X to P, but

$$\measuredangle MXY = \measuredangle MXA = \measuredangle MRA = \measuredangle MRB = \measuredangle MPB$$

so this follows.

§2 Solutions to Day 2

§2.1 JMO 2013/4, proposed by Kiran Kedlaya

Available online at https://aops.com/community/p3043748.

Problem statement

Let $f(n)$ be the number of ways to write n as a sum of powers of 2, where we keep track of the order of the summation. For example, $f(4) = 6$ because 4 can be written as 4, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Find the smallest n greater than 2013 for which $f(n)$ is odd.

The answer is 2047.

For convenience, we agree that $f(0) = 1$. Then by considering cases on the first number in the representation, we derive the recurrence

$$f(n) = \sum_{k=0}^{\lfloor \log_2 n \rfloor} f(n - 2^k). \qquad (\heartsuit)$$

We wish to understand the parity of f. The first few values are

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 1 \\ f(2) &= 2 \\ f(3) &= 3 \\ f(4) &= 6 \\ f(5) &= 10 \\ f(6) &= 18 \\ f(7) &= 31. \end{aligned}$$

Inspired by the data we make the key claim that

Claim — $f(n)$ is odd if and only if $n+1$ is a power of 2.

Proof. We call a number *repetitive* if it is zero or its binary representation consists entirely of 1's. So we want to prove that $f(n)$ is odd if and only if n is repetitive.

This only takes a few cases:

- If $n = 2^k$, then $(\heartsuit)$ has exactly two repetitive terms on the right-hand side, namely 0 and $2^k - 1$.
- If $n = 2^k + 2^\ell - 1$, then $(\heartsuit)$ has exactly two repetitive terms on the right-hand side, namely $2^{\ell+1} - 1$ and $2^\ell - 1$.
- If $n = 2^k - 1$, then $(\heartsuit)$ has exactly one repetitive terms on the right-hand side, namely $2^{k-1} - 1$.
- For other n, there are no repetitive terms at all on the right-hand side of $(\heartsuit)$.

Thus the induction checks out. □

So the final answer to the problem is 2047.

§2.2 JMO 2013/5, proposed by Zuming Feng

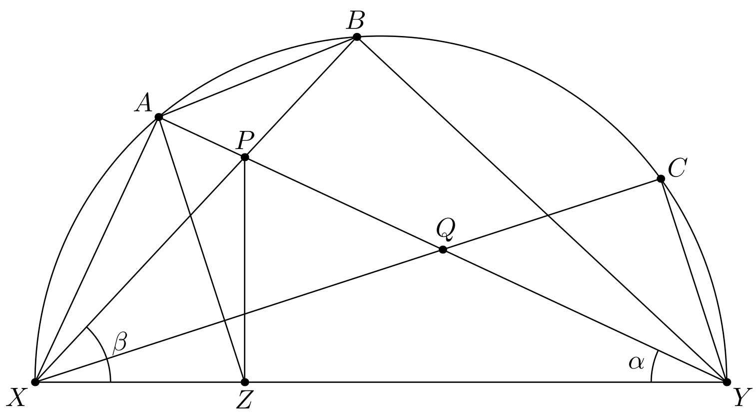
Available online at https://aops.com/community/p3043750.

Problem statement

Quadrilateral $XABY$ is inscribed in the semicircle ω with diameter $\overline{XY}$. Segments AY and BX meet at P. Point Z is the foot of the perpendicular from P to line $\overline{XY}$. Point C lies on ω such that line XC is perpendicular to line AZ. Let Q be the intersection of segments AY and XC. Prove that

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$

Let $\beta = \angle YXP$ and $\alpha = \angle PYX$ and set $XY = 1$. We do not direct angles in the following solution.



Observe that

$$\angle AZX = \angle APX = \alpha + \beta$$

since $APZX$ is cyclic. In particular, $\angle CXY = 90^\circ - (\alpha + \beta)$. It is immediate that

$$BY = \sin\beta, \quad CY = \cos(\alpha+\beta), \quad AY = \cos\alpha, \quad AX = \sin\alpha.$$

The Law of Sines on $\triangle XPY$ gives $XP = XY\frac{\sin\alpha}{\sin(\alpha+\beta)}$, and on $\triangle XQY$ gives $XQ = XY\frac{\sin\alpha}{\sin(90+\beta)} = \frac{\sin\alpha}{\cos\beta}$. So, the given is equivalent to

$$\frac{\sin\beta}{\frac{\sin\alpha}{\sin(\alpha+\beta)}} + \frac{\cos(\alpha+\beta)}{\frac{\sin\alpha}{\cos\beta}} = \frac{\cos\alpha}{\sin\alpha}$$

which is equivalent to $\cos\alpha = \cos\beta\cos(\alpha+\beta) + \sin\beta\sin(\alpha+\beta)$. This is obvious, because the right-hand side is just $\cos((\alpha+\beta)-\beta)$.

§2.3 JMO 2013/6, proposed by Titu Andreescu

Available online at https://aops.com/community/p3043752.

Problem statement

Find all real numbers $x, y, z \geq 1$ satisfying

$$\min\left(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\right) = \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

Set $x = 1+a$, $y = 1+b$, $z = 1+c$ which eliminates the $x, y, z \geq 1$ condition. Assume without loss of generality that $a \leq b \leq c$. Then the given equation rewrites as

$$\sqrt{(1+a)\left(1+(1+b)(1+c)\right)} = \sqrt{a}+\sqrt{b}+\sqrt{c}.$$

In fact, we are going to prove the left-hand side always exceeds the right-hand side, and then determine the equality cases. We have:

$$\begin{aligned}(1+a)\left(1+(1+b)(1+c)\right) &= (a+1)\left(1+(b+1)(1+c)\right) \\ &\leq (a+1)\left(1+\left(\sqrt{b}+\sqrt{c}\right)^2\right) \\ &\leq \left(\sqrt{a}+\left(\sqrt{b}+\sqrt{c}\right)\right)^2\end{aligned}$$

by two applications of Cauchy-Schwarz.

Equality holds if $bc = 1$ and $1/a = \sqrt{b}+\sqrt{c}$. Letting $c = t^2$ for $t \geq 1$, we recover $b = t^{-2} \leq t^2$ and $a = \frac{1}{t+1/t} \leq t^2$.

Hence the solution set is

$$(x, y, z) = \left(1+\left(\frac{t}{t^2+1}\right)^2, 1+\frac{1}{t^2}, 1+t^2\right)$$

and permutations, for any $t > 0$.