JMO 2012 Solution Notes

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This is an compilation of solutions for the 2012 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

2. Find all integers $n \ge 3$ such that among any n positive real numbers a_1, a_2, $\dots$, a_n with
$$\max(a_1, a_2, \dots, a_n) \le n \cdot \min(a_1, a_2, \dots, a_n),$$
there exist three that are the side lengths of an acute triangle.

3. For $a, b, c > 0$ prove that
$$\frac{a^3+3b^3}{5a+b} + \frac{b^3+3c^3}{5b+c} + \frac{c^3+3a^3}{5c+a} \ge \frac{2}{3}(a^2+b^2+c^2).$$

4. Let α be an irrational number with $0 < \alpha < 1$, and draw a circle in the plane whose circumference has length 1. Given any integer $n \ge 3$, define a sequence of points $P_1, P_2, \dots, P_n$ as follows. First select any point P_1 on the circle, and for $2 \le k \le n$ define P_k as the point on the circle for which the length of arc $P_{k-1}P_k$ is α, when travelling counterclockwise around the circle from P_{k-1} to P_k. Suppose that P_a and P_b are the nearest adjacent points on either side of P_n. Prove that $a + b \le n$.

5. For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \le k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

6. Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

§1 JMO 2012/1, proposed by Sungyoon Kim and Inseok Seo

Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Assume for contradiction that (PRS) and (QRS) are distinct. Then $\overline{RS}$ is the radical axis of these two circles. However, $\overline{AP}$ is tangent to (PRS) and $\overline{AQ}$ is tangent to (QRS), so point A has equal power to both circles, which is impossible since A does not lie on line BC.

§2 JMO 2012/2, proposed by Titu Andreescu

Find all integers $n \ge 3$ such that among any n positive real numbers $a_1, a_2, \dots, a_n$ with

$$\max(a_1, a_2, \dots, a_n) \le n \cdot \min(a_1, a_2, \dots, a_n),$$

there exist three that are the side lengths of an acute triangle.

The answer is all $n \ge 13$.

Define (F_n) as the sequence of Fibonacci numbers, by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. We will find that Fibonacci numbers show up naturally when we work through the main proof, so we will isolate the following calculation now to make the subsequent solution easier to read.

Claim — For positive integers m, we have $F_m \le m^2$ if and only if $m \le 12$.

Proof. A table of the first 14 Fibonacci numbers is given below.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
1	1	2	3	5	8	13	21	34	55	89	144	233	377

By examining the table, we see that $F_m \le m^2$ is true for $m = 1, 2, \dots 12$, and in fact $F_{12} = 12^2 = 144$. However, $F_m > m^2$ for $m = 13$ and $m = 14$.

Now it remains to prove that $F_m > m^2$ for $m \ge 15$. The proof is by induction with base cases $m = 13$ and $m = 14$ being checked already. For the inductive step, if $m \ge 15$ then we have

$$\begin{aligned} F_m &= F_{m-1} + F_{m-2} > (m-1)^2 + (m-2)^2 \\ &= 2m^2 - 6m + 5 = m^2 + (m-1)(m-5) > m^2 \end{aligned}$$

as desired. □

We now proceed to the main problem. The hypothesis $\max(a_1, a_2, \dots, a_n) \le n \cdot \min(a_1, a_2, \dots, a_n)$ will be denoted by (†).

Proof that all $n \ge 13$ have the property. We first show now that every $n \ge 13$ has the desired property. Suppose for contradiction that no three numbers are the sides of an acute triangle. Assume without loss of generality (by sorting the numbers) that $a_1 \le a_2 \le \dots \le a_n$. Then since a_{i-1}, a_i, a_{i+1} are not the sides of an acute triangle for each $i \ge 2$, we have that $a_{i+1}^2 \ge a_i^2 + a_{i-1}^2$; writing this out gives

$$\begin{aligned} a_3^2 &\ge a_2^2 + a_1^2 \ge 2a_1^2 \\ a_4^2 &\ge a_3^2 + a_2^2 \ge 2a_1^2 + a_1^2 = 3a_1^2 \\ a_5^2 &\ge a_4^2 + a_3^2 \ge 3a_1^2 + 2a_1^2 = 5a_1^2 \\ a_6^2 &\ge a_5^2 + a_4^2 \ge 5a_1^2 + 3a_1^2 = 8a_1^2 \end{aligned}$$

and so on. The Fibonacci numbers appear naturally and by induction, we conclude that $a_i^2 \ge F_i a_1^2$. In particular, $a_n^2 \ge F_n a_1^2$.

However, we know $\max(a_1, \dots, a_n) = a_n$ and $\min(a_1, \dots, a_n) = a_1$, so (†) reads $a_n \le n \cdot a_1$. Therefore we have $F_n \le n^2$, and so $n \le 12$, contradiction!

Proof that no $n \le 12$ have the property. Assume that $n \le 12$. The above calculation also suggests a way to pick the counterexample: we choose $a_i = \sqrt{F_i}$ for every i. Then $\min(a_1, \dots, a_n) = a_1 = 1$ and $\max(a_1, \dots, a_n) = \sqrt{F_n}$, so (†) is true as long as $n \le 12$. And indeed no three numbers form the sides of an acute triangle: if $i < j < k$, then $a_k^2 = F_k = F_{k-1} + F_{k-2} \ge F_j + F_i = a_j^2 + a_i^2$.

§3 JMO 2012/3, proposed by Titu Andreescu

For $a, b, c > 0$ prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geq \frac{2}{3}(a^2 + b^2 + c^2).$$

Apply Titu lemma to get

$$\sum_{\text{cyc}} \frac{a^3}{5a + b} = \sum_{\text{cyc}} \frac{a^4}{5a^2 + ab} \geq \frac{(a^2 + b^2 + c^2)^2}{\sum_{\text{cyc}}(5a^2 + ab)} \geq \frac{a^2 + b^2 + c^2}{6}$$

where the last step follows from the identity $\sum_{\text{cyc}}(5a^2 + ab) \leq 6(a^2 + b^2 + c^2)$.

Similarly,

$$\sum_{\text{cyc}} \frac{b^3}{5a + b} = \sum_{\text{cyc}} \frac{b^4}{5ab + b^2} \geq \frac{(a^2 + b^2 + c^2)^2}{\sum_{\text{cyc}}(5ab + b^2)} \geq \frac{a^2 + b^2 + c^2}{6}$$

using the fact that $\sum_{\text{cyc}} 5ab + b^2 \leq 6(a^2 + b^2 + c^2)$.

Therefore, adding the first display to three times the second display implies the result.

§4 JMO 2012/4, proposed by Sam Vandervelde

Let α be an irrational number with $0 < \alpha < 1$, and draw a circle in the plane whose circumference has length 1. Given any integer $n \geq 3$, define a sequence of points $P_1, P_2, \dots, P_n$ as follows. First select any point P_1 on the circle, and for $2 \leq k \leq n$ define P_k as the point on the circle for which the length of arc $P_{k-1}P_k$ is α, when travelling counterclockwise around the circle from P_{k-1} to P_k. Suppose that P_a and P_b are the nearest adjacent points on either side of P_n. Prove that $a + b \leq n$.

No points coincide since α is irrational.

Assume for contradiction that $n < a + b < 2n$. Then

$$\overline{P_n P_{a+b-n}} \parallel \overline{P_a P_b}.$$

This is an obvious contradiction since then P_{a+b-n} is contained in the arc $\widehat{P_a P_b}$ of the circle through P_n.

§5 JMO 2012/5, proposed by Warut Suksompong

For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \le k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

The answer is $S = 502$ (not 503!).

Claim — If $\gcd(k, 2012) = 1$, then necessarily either k or $2012 - k$ will counts towards S.

Proof. First note that both ak, bk are nonzero modulo 2012. Note also that $ak \not\equiv bk \pmod{2012}$.

So if r_a is the remainder of $ak \pmod{2012}$, then $2012 - r_a$ is the remainder of $a(2012-k) \pmod{2012}$ Similarly we can consider r_b and $2012 - r_b$. As mentioned already, we have $r_a \neq r_b$. So either $r_a > r_b$ or $2012 - r_a > 2012 - r_b$. □

This implies $S \ge \frac{1}{2}\varphi(2012) = 502$.

But this can actually be achieved by taking $a = 4$ and $b = 1010$, since

- If k is even, then $ak \equiv bk \pmod{2012}$ so no even k counts towards S; and
- If $k \equiv 0 \pmod{503}$, then $ak \equiv 0 \pmod{2012}$ so no such k counts towards S.

This gives the final answer $S \ge 502$.

Remark. A similar proof works with 2012 replaced by any n and will give an answer of $\frac{1}{2}\varphi(n)$. For composite n, one uses the Chinese remainder theorem to pick distinct a and b not divisible by n such that $\operatorname{lcm}(a-b, a) = n$.

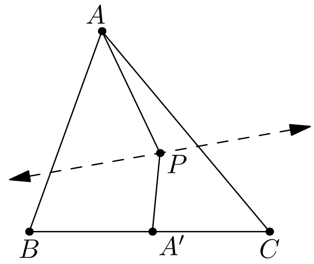
§6 JMO 2012/6, proposed by Titu Andreescu and Cosmin Pohoata

Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

We present two solutions.

First solution (complex numbers) Let $p = 0$ and set γ as the real line. Then A' is the intersection of bc and $p\overline{a}$. So, we get

$$a' = \frac{\overline{a}(\overline{b}c - b\overline{c})}{(\overline{b} - \overline{c})\overline{a} - (b-c)a}.$$



Note that

$$\overline{a}' = \frac{a(b\overline{c} - \overline{b}c)}{(b-c)a - (\overline{b} - \overline{c})\overline{a}}.$$

Thus it suffices to prove

$$0 = \begin{vmatrix} \frac{\overline{a}(\overline{b}c - b\overline{c})}{(\overline{b}-\overline{c})\overline{a}-(b-c)a} & \frac{a(b\overline{c}-\overline{b}c)}{(b-c)a-(\overline{b}-\overline{c})\overline{a}} & 1 \\ \frac{\overline{b}(\overline{c}a-c\overline{a})}{(\overline{c}-\overline{a})\overline{b}-(c-a)b} & \frac{b(c\overline{a}-\overline{c}a)}{(c-a)b-(\overline{c}-\overline{a})\overline{b}} & 1 \\ \frac{\overline{c}(\overline{a}b-a\overline{b})}{(\overline{a}-\overline{b})\overline{c}-(a-b)c} & \frac{c(a\overline{b}-\overline{a}b)}{(a-b)c-(\overline{a}-\overline{b})\overline{c}} & 1 \end{vmatrix}.$$

This is equivalent to

$$0 = \begin{vmatrix} \overline{a}(\overline{b}c - b\overline{c}) & a(\overline{b}c - b\overline{c}) & (\overline{b}-\overline{c})\overline{a} - (b-c)a \\ \overline{b}(\overline{c}a - c\overline{a}) & b(\overline{c}a - c\overline{a}) & (\overline{c}-\overline{a})\overline{b} - (c-a)b \\ \overline{c}(\overline{a}b - a\overline{b}) & c(\overline{a}b - a\overline{b}) & (\overline{a}-\overline{b})\overline{c} - (a-b)c \end{vmatrix}.$$

Evaluating the determinant gives

$$\sum_{\text{cyc}} ((\overline{b} - \overline{c})\overline{a} - (b-c)a) \cdot - \begin{vmatrix} b & \overline{b} \\ c & \overline{c} \end{vmatrix} \cdot (\overline{c}a - c\overline{a}) \left(\overline{a}b - a\overline{b}\right)$$

or, noting the determinant is $b\overline{c} - \overline{b}c$ and factoring it out,

$$(\overline{b}c - c\overline{b})(\overline{c}a - c\overline{a})(\overline{a}b - a\overline{b}) \sum_{\text{cyc}} \left(ab - ac + \overline{ca} - \overline{ba}\right) = 0.$$

Second solution (Desargues involution) We let $C'' = \overline{A'B'} \cap \overline{AB}$. Consider complete quadrilateral $ABCA'B'C''C$. We see that there is an involutive pairing τ at P swapping (PA, PA'), (PB, PB'), (PC, PC''). From the first two, we see τ coincides with reflection about ℓ, hence conclude $C'' = C$.