# JMO 2012 Solution Notes

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This is a compilation of solutions for the 2012 JMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let  $\mathbb{R}$  denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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## §0 Problems

- 1. Given a triangle ABC, let P and Q be points on segments  $\overline{AB}$  and  $\overline{AC}$ , respectively, such that AP = AQ. Let S and R be distinct points on segment  $\overline{BC}$  such that S lies between B and R,  $\angle BPS = \angle PRS$ , and  $\angle CQR = \angle QSR$ . Prove that P, Q, R, S are concyclic.
- **2.** Find all integers  $n \ge 3$  such that among any n positive real numbers  $a_1, a_2, \ldots, a_n$  with

$$\max(a_1, a_2, \ldots, a_n) \le n \cdot \min(a_1, a_2, \ldots, a_n),$$

there exist three that are the side lengths of an acute triangle.

**3.** For a, b, c > 0 prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \ge \frac{2}{3}(a^2 + b^2 + c^2).$$

- 4. Let  $\alpha$  be an irrational number with  $0 < \alpha < 1$ , and draw a circle in the plane whose circumference has length 1. Given any integer  $n \ge 3$ , define a sequence of points  $P_1, P_2, \ldots, P_n$  as follows. First select any point  $P_1$  on the circle, and for  $2 \le k \le n$  define  $P_k$  as the point on the circle for which the length of arc  $P_{k-1}P_k$  is  $\alpha$ , when travelling counterclockwise around the circle from  $P_{k-1}$  to  $P_k$ . Suppose that  $P_a$  and  $P_b$  are the nearest adjacent points on either side of  $P_n$ . Prove that  $a + b \le n$ .
- 5. For distinct positive integers a, b < 2012, define f(a, b) to be the number of integers k with  $1 \le k < 2012$  such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of f(a, b), where a and b range over all pairs of distinct positive integers less than 2012. Determine S.
- 6. Let P be a point in the plane of  $\triangle ABC$ , and  $\gamma$  a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to  $\gamma$  intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

# §1 Solutions to Day 1

## §1.1 JMO 2012/1, proposed by Sungyoon Kim, Inseok Seo

Available online at https://aops.com/community/p2669111.

## **Problem statement**

Given a triangle ABC, let P and Q be points on segments  $\overline{AB}$  and  $\overline{AC}$ , respectively, such that AP = AQ. Let S and R be distinct points on segment  $\overline{BC}$  such that S lies between B and R,  $\angle BPS = \angle PRS$ , and  $\angle CQR = \angle QSR$ . Prove that P, Q, R, S are concyclic.

Assume for contradiction that (PRS) and (QRS) are distinct. Then  $\overline{RS}$  is the radical axis of these two circles. However,  $\overline{AP}$  is tangent to (PRS) and  $\overline{AQ}$  is tangent to (QRS), so point A has equal power to both circles, which is impossible since A does not lie on line BC.

#### §1.2 JMO 2012/2, proposed by Titu Andreescu

Available online at https://aops.com/community/p2669112.

#### **Problem statement**

Find all integers  $n \ge 3$  such that among any n positive real numbers  $a_1, a_2, \ldots, a_n$  with

 $\max(a_1, a_2, \ldots, a_n) \le n \cdot \min(a_1, a_2, \ldots, a_n),$ 

there exist three that are the side lengths of an acute triangle.

The answer is all  $n \ge 13$ .

Define  $(F_n)$  as the sequence of Fibonacci numbers, by  $F_1 = F_2 = 1$  and  $F_{n+1} = F_n + F_{n-1}$ . We will find that Fibonacci numbers show up naturally when we work through the main proof, so we will isolate the following calculation now to make the subsequent solution easier to read.

**Claim** — For positive integers m, we have  $F_m \leq m^2$  if and only if  $m \leq 12$ .

*Proof.* A table of the first 14 Fibonacci numbers is given below.

By examining the table, we see that  $F_m \leq m^2$  is true for m = 1, 2, ..., 12, and in fact  $F_{12} = 12^2 = 144$ . However,  $F_m > m^2$  for m = 13 and m = 14.

Now it remains to prove that  $F_m > m^2$  for  $m \ge 15$ . The proof is by induction with base cases m = 13 and m = 14 being checked already. For the inductive step, if  $m \ge 15$  then we have

$$F_m = F_{m-1} + F_{m-2} > (m-1)^2 + (m-2)^2$$
  
= 2m<sup>2</sup> - 6m + 5 = m<sup>2</sup> + (m-1)(m-5) > m<sup>2</sup>

as desired.

We now proceed to the main problem. The hypothesis  $\max(a_1, a_2, \ldots, a_n) \leq n \cdot \min(a_1, a_2, \ldots, a_n)$  will be denoted by (†).

**Proof that all**  $n \ge 13$  have the property. We first show now that every  $n \ge 13$  has the desired property. Suppose for contradiction that no three numbers are the sides of an acute triangle. Assume without loss of generality (by sorting the numbers) that  $a_1 \le a_2 \le \cdots \le a_n$ . Then since  $a_{i-1}$ ,  $a_i$ ,  $a_{i+1}$  are not the sides of an acute triangle for each  $i \ge 2$ , we have that  $a_{i+1}^2 \ge a_i^2 + a_{i-1}^2$ ; writing this out gives

$$\begin{aligned} a_3^2 &\geq a_2^2 + a_1^2 \geq 2a_1^2 \\ a_4^2 &\geq a_3^2 + a_2^2 \geq 2a_1^2 + a_1^2 = 3a_1^2 \\ a_5^2 &\geq a_4^2 + a_3^2 \geq 3a_1^2 + 2a_1^2 = 5a_1^2 \\ a_6^2 &\geq a_5^2 + a_4^2 \geq 5a_1^2 + 3a_1^2 = 8a_1^2 \end{aligned}$$

and so on. The Fibonacci numbers appear naturally and by induction, we conclude that  $a_i^2 \ge F_i a_1^2$ . In particular,  $a_n^2 \ge F_n a_1^2$ .

However, we know  $\max(a_1, \ldots, a_n) = a_n$  and  $\min(a_1, \ldots, a_n) = a_1$ , so (†) reads  $a_n \leq n \cdot a_1$ . Therefore we have  $F_n \leq n^2$ , and so  $n \leq 12$ , contradiction!

**Proof that no**  $n \leq 12$  have the property. Assume that  $n \leq 12$ . The above calculation also suggests a way to pick the counterexample: we choose  $a_i = \sqrt{F_i}$  for every *i*. Then  $\min(a_1, \ldots, a_n) = a_1 = 1$  and  $\max(a_1, \ldots, a_n) = \sqrt{F_n}$ , so  $(\dagger)$  is true as long as  $n \leq 12$ . And indeed no three numbers form the sides of an acute triangle: if i < j < k, then  $a_k^2 = F_k = F_{k-1} + F_{k-2} \geq F_j + F_i = a_j^2 + a_i^2$ .

## §1.3 JMO 2012/3, proposed by Titu Andreescu

Available online at https://aops.com/community/p2669114.

**Problem statement** 

For a, b, c > 0 prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \ge \frac{2}{3}(a^2 + b^2 + c^2).$$

Here are two possible approaches.

**Cauchy-Schwarz approach.** Apply Titu lemma to get

$$\sum_{\text{cyc}} \frac{a^3}{5a+b} = \sum_{\text{cyc}} \frac{a^4}{5a^2+ab} \ge \frac{(a^2+b^2+c^2)^2}{\sum_{\text{cyc}}(5a^2+ab)} \ge \frac{a^2+b^2+c^2}{6}$$

where the last step follows from the identity  $\sum_{cyc} (5a^2 + ab) \le 6(a^2 + b^2 + c^2)$ . Similarly,

$$\sum_{\text{cyc}} \frac{b^3}{5a+b} = \sum_{\text{cyc}} \frac{b^4}{5ab+b^2} \ge \frac{(a^2+b^2+c^2)^2}{\sum_{\text{cyc}}(5ab+b^2)} \ge \frac{a^2+b^2+c^2}{6}$$

using the fact that  $\sum_{\text{cyc}} 5ab + b^2 \leq 6(a^2 + b^2 + c^2)$ . Therefore, adding the first display to three times the second display implies the result.

**¶ Cauchy-Schwarz approach.** The main magical claim is:

Claim — We have

$$\frac{a^3 + 3b^3}{5a+b} \ge \frac{25}{36}b^2 - \frac{1}{36}a^2.$$

*Proof.* Let x = a/b > 0. The desired inequality is equivalent to

$$\frac{x^3+3}{5x+1} \ge \frac{25-x^2}{36}.$$

However,

$$36(x^{3}+3) - (5x+1)(25-x^{2}) = 41x^{3} + x^{2} - 125x + 83$$
$$= (x-1)^{2}(41x+83) \ge 0.$$

Sum the claim cyclically to finish.

**Remark** (Derivation of the main claim). The overall strategy is to hope for a constant ksuch that

$$\frac{a^3 + 3b^3}{5a + b} \ge ka^2 + \left(\frac{2}{3} - k\right)b^2$$

is true. Letting x = a/b as above and expanding, we need a value k such that the cubic polynomial

$$P(x) := (x^3 + 3) - (5x + 1)\left(kx^2 + \left(\frac{2}{3} - k\right)\right) = (1 - 5k)x^3 - kx^2 + \left(5k - \frac{10}{3}\right)x + \left(k + \frac{7}{3}\right)$$

is nonnegative everywhere. Since P(1) = 0 necessarily, in order for  $P(1-\varepsilon)$  and  $P(1+\varepsilon)$  to both be nonnegative (for small  $\varepsilon$ ), the polynomial P must have a double root at 1, meaning the first derivative P'(1) = 0 needs to vanish. In other words, we need

$$3(1-5k) - 2k + \left(5k - \frac{10}{3}\right) = 0.$$

Solving gives k = -1/36. One then factors out the repeated root  $(x - 1)^2$  from the resulting P.

# §2 Solutions to Day 2

# §2.1 JMO 2012/4, proposed by Sam Vandervelde

Available online at https://aops.com/community/p2669956.

#### **Problem statement**

Let  $\alpha$  be an irrational number with  $0 < \alpha < 1$ , and draw a circle in the plane whose circumference has length 1. Given any integer  $n \ge 3$ , define a sequence of points  $P_1, P_2, \ldots, P_n$  as follows. First select any point  $P_1$  on the circle, and for  $2 \le k \le n$ define  $P_k$  as the point on the circle for which the length of arc  $P_{k-1}P_k$  is  $\alpha$ , when travelling counterclockwise around the circle from  $P_{k-1}$  to  $P_k$ . Suppose that  $P_a$  and  $P_b$  are the nearest adjacent points on either side of  $P_n$ . Prove that  $a + b \le n$ .

No points coincide since  $\alpha$  is irrational.

Assume for contradiction that n < a + b < 2n. Then it follows that

$$\overline{P_n P_{a+b-n}} \parallel \overline{P_a P_b}$$

as shown below.



This is an obvious contradiction since then  $P_{a+b-n}$  is contained in the arc  $\widehat{P_aP_b}$  of the circle through  $P_n$ .

#### §2.2 JMO 2012/5, proposed by Warut Suksompong

Available online at https://aops.com/community/p2669967.

#### **Problem statement**

For distinct positive integers a, b < 2012, define f(a, b) to be the number of integers k with  $1 \le k < 2012$  such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of f(a, b), where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

The answer is  $S = 502 \pmod{503!}$ .

**Claim** — If gcd(k, 2012) = 1, then necessarily either k or 2012 - k will counts towards S.

*Proof.* First note that both ak, bk are nonzero modulo 2012. Note also that  $ak \neq bk$  (mod 2012).

So if  $r_a$  is the remainder of  $ak \pmod{2012}$ , then  $2012-r_a$  is the remainder of  $a(2012-k) \pmod{2012}$  Similarly we can consider  $r_b$  and  $2012-r_b$ . As mentioned already, we have  $r_a \neq r_b$ . So either  $r_a > r_b$  or  $2012 - r_a > 2012 - r_b$ .

This implies  $S \ge \frac{1}{2}\varphi(2012) = 502.$ 

But this can actually be achieved by taking a = 4 and b = 1010, since

- If k is even, then  $ak \equiv bk \pmod{2012}$  so no even k counts towards S; and
- If  $k \equiv 0 \pmod{503}$ , then  $ak \equiv 0 \pmod{2012}$  so no such k counts towards S.

This gives the final answer  $S \ge 502$ .

**Remark.** A similar proof works with 2012 replaced by any n and will give an answer of  $\frac{1}{2}\varphi(n)$ . For composite n, one uses the Chinese remainder theorem to pick distinct a and b not divisible by n such that  $\operatorname{lcm}(a - b, a) = n$ .

### §2.3 JMO 2012/6, proposed by Titu Andreescu, Cosmin Pohoata

Available online at https://aops.com/community/p2669960.

**Problem statement** 

Let P be a point in the plane of  $\triangle ABC$ , and  $\gamma$  a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to  $\gamma$  intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

We present three solutions.

¶ First solution (complex numbers). Let p = 0 and set  $\gamma$  as the real line. Then A' is the intersection of bc and  $p\overline{a}$ . So, we get



Note that

$$\overline{a}' = \frac{a(b\overline{c} - bc)}{(b - c)a - (\overline{b} - \overline{c})\overline{a}}.$$

Thus it suffices to prove

$$0 = \det \begin{bmatrix} \frac{\overline{a}(bc-b\overline{c})}{(\overline{b}-\overline{c})\overline{a}-(b-c)a} & \frac{a(b\overline{c}-bc)}{(b-c)a-(\overline{b}-\overline{c})\overline{a}} & 1\\ \frac{\overline{b}(\overline{c}a-c\overline{a})}{(\overline{c}-\overline{a})\overline{b}-(c-a)b} & \frac{b(c\overline{a}-\overline{c}a)}{(c-a)b-(\overline{c}-\overline{a})\overline{b}} & 1\\ \frac{\overline{c}(\overline{a}b-a\overline{b})}{(\overline{a}-\overline{b})\overline{c}-(a-b)c} & \frac{c(a\overline{b}-\overline{a}b)}{(a-b)c-(\overline{a}-\overline{b})\overline{c}} & 1 \end{bmatrix}.$$

This is equivalent to

$$0 = \det \begin{bmatrix} \overline{a}(\overline{b}c - b\overline{c}) & a(\overline{b}c - b\overline{c}) & (\overline{b} - \overline{c})\overline{a} - (b - c)a\\ \overline{b}(\overline{c}a - c\overline{a}) & b(\overline{c}a - c\overline{a}) & (\overline{c} - \overline{a})\overline{b} - (c - a)b\\ \overline{c}(\overline{a}b - a\overline{b}) & c(\overline{a}b - a\overline{b}) & (\overline{a} - \overline{b})\overline{c} - (a - b)c \end{bmatrix}.$$

This determinant has the property that the rows sum to zero, and we're done.

Remark. Alternatively, if you don't notice that you could just blindly expand:

$$\sum_{cyc} ((\overline{b} - \overline{c})\overline{a} - (b - c)a) \cdot -\det \begin{bmatrix} b & \overline{b} \\ c & \overline{c} \end{bmatrix} (\overline{c}a - c\overline{a}) (\overline{a}b - a\overline{b})$$
$$= (\overline{b}c - c\overline{b})(\overline{c}a - c\overline{a})(\overline{a}b - a\overline{b}) \sum_{cyc} (ab - ac + \overline{c}\overline{a} - \overline{b}\overline{a}) = 0.$$

¶ Second solution (Desargues involution). We let  $C'' = \overline{A'B'} \cap \overline{AB}$ . Consider complete quadrilateral ABCA'B'C''C. We see that there is an involutive pairing  $\tau$  at P swapping (PA, PA'), (PB, PB'), (PC, PC''). From the first two, we see  $\tau$  coincides with reflection about  $\ell$ , hence conclude C'' = C.

¶ Third solution (barycentric), by Catherine Xu. We will perform barycentric coordinates on the triangle PCC', with P = (1,0,0), C' = (0,1,0), and C = (0,0,1). Set a = CC', b = CP, c = C'P as usual. Since A, B, C' are collinear, we will define A = (p: k: q) and B = (p: l: q).

**Claim** — Line  $\gamma$  is the angle bisector of  $\angle APA'$ ,  $\angle BPB'$ , and  $\angle CPC'$ .

*Proof.* Since A'P is the reflection of AP across  $\gamma$ , etc.

Thus B' is the intersection of the isogonal of B with respect to  $\angle P$  with the line CA; that is,

$$B' = \left(\frac{p}{k}\frac{b^2}{\ell}:\frac{b^2}{\ell}:\frac{c^2}{q}\right).$$

Analogously, A' is the intersection of the isogonal of A with respect to  $\angle P$  with the line CB; that is,

$$A' = \left(\frac{p}{\ell}\frac{b^2}{k}:\frac{b^2}{k}:\frac{c^2}{q}\right).$$

The ratio of the first to third coordinate in these two points is both  $b^2pq : c^2k\ell$ , so it follows A', B', and C' are collinear.

**Remark** (Problem reference). The converse of this problem appears as problem 1052 attributed S. V. Markelov in the book *Geometriya: 9–11 Klassy: Ot Uchebnoy Zadachi k Tvorcheskoy, 1996*, by I. F. Sharygin.