IMO 2026 Solution Notes

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17 July 2026

This is a compilation of solutions for the 2026 IMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. There are 2026 integers greater than 1 written on a blackboard, not necessarily different. In a move, Confucius chooses two integers $m > 1$ and $n > 1$ from different places on the blackboard and replaces these two integers with
$$\gcd(m, n) \quad \text{and} \quad \frac{\operatorname{lcm}(m, n)}{\gcd(m, n)}.$$
He continues to make moves while it is possible to do so.
   (a) Prove that, regardless of the choices of Confucius, after finitely many moves, exactly one integer M on the blackboard is greater than 1.
   (b) Prove that the value of M does not depend on the choices of Confucius.

2. Let ABC be a triangle and let points M and N be the midpoints of sides AB and AC, respectively. Let points K and L be chosen inside triangles BMC and BNC, respectively, such that K lies inside the angle LBA, L lies inside the angle ACK, and
$$\angle KBA = \angle ACL, \quad \angle LBK = \angle LNC, \quad \angle LCK = \angle BMK.$$
Let O be the circumcenter of triangle AKL. Prove that $OM = ON$.

3. Let n be a positive integer. Liu Bang and Xiang Yu have a stick of length 1 and want to divide it between themselves. Liu Bang marks at most n points on the stick, and then Xiang Yu marks at most n points on the stick. The marked points are distinct. Then, the stick is cut at all marked points, creating a number of pieces. Afterwards, they take turns claiming any unclaimed piece of the stick, with Liu Bang going first. Each player's goal is to maximise the total length of their own pieces. For each n, determine the largest value c such that Liu Bang may guarantee a total length of at least c, regardless of Xiang Yu's play.

4. Shan-Yu and Mulan are playing a game. Let θ be an angle with $0^\circ < \theta < 180^\circ$ known to both players. Initially, Shan-Yu makes a paper triangle $\mathcal{T}$ with measurements of his choice. Then, they repeatedly perform the following steps:
   - If $\mathcal{T}$ has at least one angle measuring exactly θ, then the game stops and Mulan wins.
   - Otherwise, Mulan chooses a point P on the perimeter of $\mathcal{T}$, different from its three vertices. She then makes a straight cut from P to the opposite vertex of $\mathcal{T}$, splitting it into two triangles.
   - Shan-Yu discards one of the two triangles. The remaining triangle becomes the new $\mathcal{T}$.

   For which real values of θ can Mulan guarantee her victory in finitely many steps, no matter how Shan-Yu plays?

5. Solve over $\mathbb{R}_{>0}$ the functional equation
$$\sqrt{\frac{x^2 + f(y)^2}{2}} \geq \frac{f(x) + y}{2} \geq \sqrt{x f(y)}.$$

6. Let $a_1, a_2, a_3, \dots$ be an infinite sequence of positive integers greater than 1. Suppose that for all positive integers n, the number a_{n+1} is the smallest positive integer

greater than a_n such that $\gcd(a_{n+1}, a_i) > 1$ for every $i = 1, 2, \dots, n$. Prove that there exist positive integers T and L such that

$$a_{n+T} = a_n + L$$

for every positive integer n.

§1 Solutions to Day 1

§1.1 IMO 2026/1, proposed by Giancarlo Kerg (LUX)

Available online at https://aops.com/community/p38664896.

Problem statement

There are 2026 integers greater than 1 written on a blackboard, not necessarily different. In a move, Confucius chooses two integers $m > 1$ and $n > 1$ from different places on the blackboard and replaces these two integers with

$$\gcd(m, n) \quad \text{and} \quad \frac{\operatorname{lcm}(m, n)}{\gcd(m, n)}.$$

He continues to make moves while it is possible to do so.

(a) Prove that, regardless of the choices of Confucius, after finitely many moves, exactly one integer M on the blackboard is greater than 1.

(b) Prove that the value of M does not depend on the choices of Confucius.

For part (a), note that a move either

- permanently increases the number of 1's on the board if $\gcd(m, n) = 1$; or
- decreases the product of all numbers on the board if $\gcd(m, n) > 1$.

This already implies the result.

For part (b), fix a prime p; it suffices to show $\nu_p(M)$ is independent of the order of the moves. But the point is this:

Claim — If the numbers on the board at some point are $t_1, \dots, t_{2026}$, then

$$\gcd(\nu_p(t_1), \nu_p(t_2), \dots, \nu_p(t_{2026}))$$

does not change under any of the possible operations.

Proof. Suppose m and n are two numbers we operate on, and $\nu_p(m) = x$, $\nu_p(n) = y$, with $x \le y$. Then $\nu_p(\gcd(m, n)) = x$ and $\nu_p(\operatorname{lcm}(m, n)/\gcd(m, n)) = y - x$. And since $\gcd(x, y - x) = \gcd(x, y)$, the result follows. □

> **Remark.** This is essentially a 2026-number Euclidean algorithm.

In particular, if the starting numbers on the board are $a_1, \dots, a_{2026}$, then

$$\nu_p(M) = \gcd(\nu_p(a_1), \nu_p(a_2), \dots, \nu_p(a_{2026}))$$

independent of the moves made.

§1.2 IMO 2026/2, proposed by Mykhailo Shtandenko (UKR)

Available online at https://aops.com/community/p38664791.

Problem statement

Let ABC be a triangle and let points M and N be the midpoints of sides AB and AC, respectively. Let points K and L be chosen inside triangles BMC and BNC, respectively, such that K lies inside the angle LBA, L lies inside the angle ACK, and

$$\angle KBA = \angle ACL, \quad \angle LBK = \angle LNC, \quad \angle LCK = \angle BMK.$$

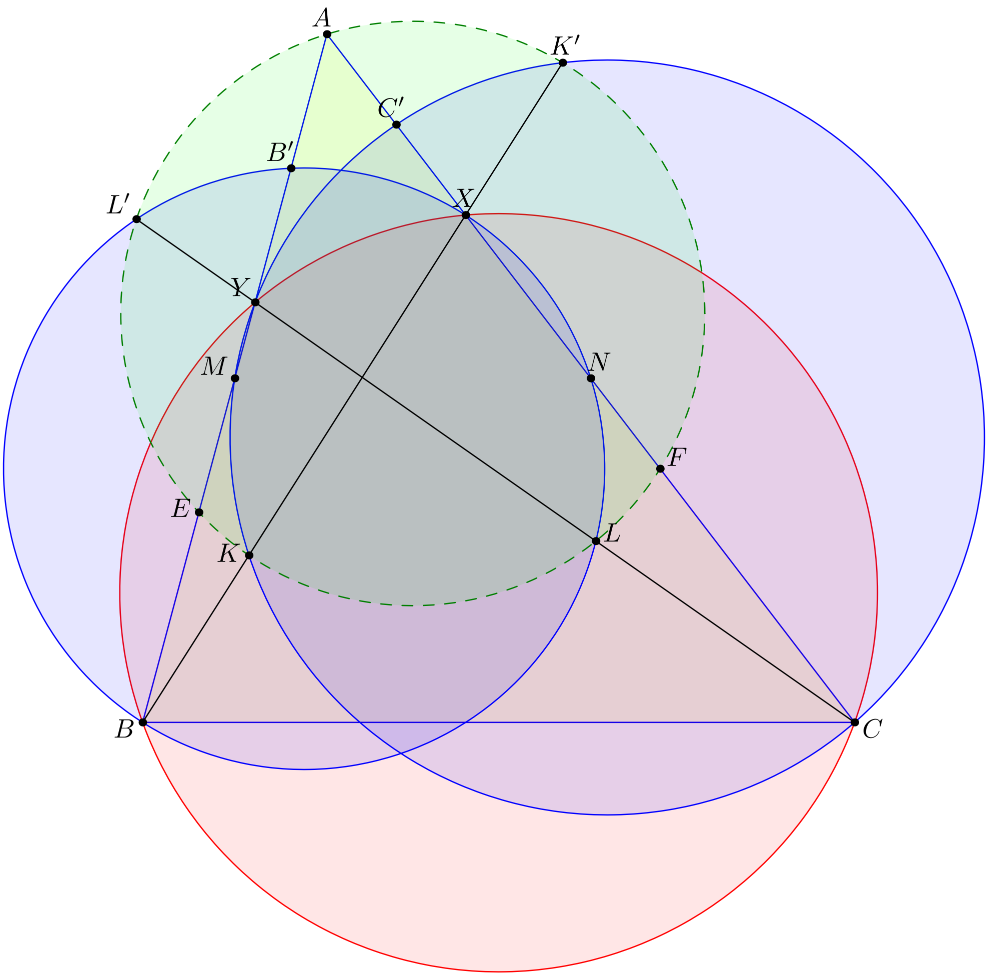
Let O be the circumcenter of triangle AKL. Prove that $OM = ON$.

This solution was shown to me by Andrew Gu and Kevin Sun. We start by defining

$$X := \overline{BK} \cap \overline{AC}, \qquad Y := \overline{CL} \cap \overline{AB}.$$

This lets us translate every angle condition in the problem into a cyclic quadrilateral:

- $\angle KBA = \angle ACL$ just means $BYXC$ is cyclic; equivalently (since $\overline{MN} \parallel \overline{BC}$) this means $MYXN$ is cyclic.
- $\angle LBK = \angle LNC$ just means $XNLB$ is cyclic; we denote this circle by ω_B.
- $\angle LCK = \angle BMK$ just means $YMKC$ is cyclic; we denote this circle by ω_C.



For the remainder of the solution we will only use the power of a point theorem repeatedly. (To avoid configuration dependence, we adopt a signed length convention that $PQ \cdot PQ' > 0$ if P does not lie inside segment $\overline{QQ'}$, and is negative otherwise.)

We let lines XK and XC meet ω_C again at K' and C'. Define L' and B' similarly on ω_B.

Claim — The points B' and C' are the midpoints of $\overline{AY}$ and $\overline{AX}$.

Proof. Since $AC' \cdot AC = AY \cdot AM = AX \cdot AN = AX \cdot \frac{1}{2}AC$, we get $AC' = \frac{1}{2}AX$. Similarly $AB' = \frac{1}{2}AY$. □

The heart of the proof comes when we introduce the midpoints E and F of $\overline{BY}$ and $\overline{CX}$ and claim:

Claim — The points A, K, L, E, F, K', L' are all concyclic.

Proof. Note that:

- $AKFK'$ is cyclic since $XK \cdot XK' = XC' \cdot XC = XA \cdot XF$.
- $AEKK'$ is cyclic since $BK \cdot BK' = BM \cdot BY = BE \cdot BA$.

Hence K and K' lie on the circumcircle of triangle AEF. Similarly, so do L and L'. □

To finish, we use the following result:

Claim — The powers of M and N with respect to (AEF) are equal. That is, $MA \cdot ME = NA \cdot NF$.

Proof. Note that $AM \cdot ME = \frac{1}{2}AM \cdot AY = \frac{1}{2}AN \cdot AX = AN \cdot NF$. □

Letting ρ denote the circumradius of $\triangle AKL$ (equivalently $\triangle AEF$), this means $OM^2 - \rho^2 = ON^2 - \rho^2$, ergo $OM = ON$, as needed.

§1.3 IMO 2026/3, proposed by RUS

Available online at https://aops.com/community/p38664789.

Problem statement

Let n be a positive integer. Liu Bang and Xiang Yu have a stick of length 1 and want to divide it between themselves. Liu Bang marks at most n points on the stick, and then Xiang Yu marks at most n points on the stick. The marked points are distinct. Then, the stick is cut at all marked points, creating a number of pieces. Afterwards, they take turns claiming any unclaimed piece of the stick, with Liu Bang going first. Each player's goal is to maximise the total length of their own pieces. For each n, determine the largest value c such that Liu Bang may guarantee a total length of at least c, regardless of Xiang Yu's play.

The answer is

$$c_n := \frac{2^n}{2^{n+1}-1},$$

which we'll show can be obtained if Liu divides the stick in ratio $1 : 2 : 4 : \cdots : 2^n$. The following solution uses some different ideas from the various shortlisted official solutions.

¶ Setup. We first make cosmetic simplifications showing some parts of the problem are irrelevant. (These parts were likely used to make the statement compact or easier to translate.)

- First, we require each player to make exactly n cuts but instead allow the marked points to coincide (giving length 0 segments).
- Also, we remove the turn-based claiming part of the problem by noting that if $x_1 \ge x_2 \ge \cdots \ge x_{2n+1} \ge 0$ are the obtained stick lengths, it's clear that Liu's score is

  $$x_1 + x_3 + x_5 + \cdots + x_{2n+1}$$

  under optimal play. (On each turn, a player should always take the largest piece available.)
- Rather than discussing Liu's score directly, we will often prefer to work with the *gap* defined as Liu's score minus Xiang's score:

  $$G := x_1 - x_2 + x_3 - x_4 + \cdots + x_{2n+1}.$$

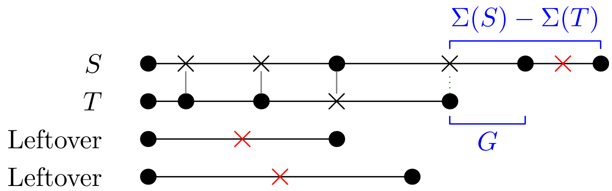
We show a strategy for Xiang that guarantees $G \le \frac{1}{2^{n+1}-1}$ regardless of how Liu starts the initial division. Then we prove that in the $1 : 2 : 4 : \cdots : 2^n$ division mentioned, we always have $G \ge \frac{1}{2^{n+1}-1}$.

By a *segment* we refer to the $n+1$ intervals (possibly of length 0) that Liu's n points divide the stick into. If S is a set of segments, we let $\Sigma(S)$ denote the sum of the lengths of the segments.

¶ Strategy for Xiang. The basic strategy is the following:

Claim — Let S and T be any two disjoint distinct subsets of the $n+1$ segments. Then Xiang can guarantee
$$G \le |\Sigma(S) - \Sigma(T)|.$$

Proof. WLOG assume $\Sigma(S) \ge \Sigma(T)$. Line up S and T as shown in the figure below and have Xiang make mirrored cuts so that we get several pairs of equal segments plus a leftover part of length $\Sigma(S) - \Sigma(T)$. Then for all the remaining segments not in S or T, Xiang bisects them. Xiang also bisects any segment in S contained in the leftover part.



As the pairs of segments of the same length cancel off, we find G is exactly equal to the length of the first segment in the overhang of S over T (as illustrated above). Since $G \le \Sigma(S) - \Sigma(T)$, the proof is complete. $\square$

Remark (When $T \ne \varnothing$, one fewer cut is needed by Xiang)**.** In the original version of the problem where we allow fewer than n cuts, then in fact Xiang may make one fewer cut: the rightmost cut of T on S is not necessary. (This is drawn with a dotted line above.) Interestingly, this means that except when $T = \varnothing$, Xiang's optimal strategy requires only $n-1$ cuts; the ability to make an n^{th} cut does not help.

The problem now follows by a standard Pigeonhole argument: consider all 2^{n+1} possible $\Sigma(S)$, which lie in the interval $[0,1]$. By pigeonhole principle there should be $S \ne T$ already such that
$$0 \le \Sigma(S) - \Sigma(T) \le \frac{1}{2^{n+1}-1}.$$
We then prune S and T further by removing any common elements, which does not change $\Sigma(S) - \Sigma(T)$.

Remark (Equality case)**.** By analyzing the equality case of the pigeonhole argument, it follows that the $1 : 2 : 4 : \cdots : 2^n$ division is the unique optimal starting move for Liu. That is, if Liu picks a different division then Xiang can ensure $G < \frac{1}{2^{n+1}-1}$.

Remark (Common failed approach)**.** A common proposed class of strategy goes something like the following. Xiang runs an algorithm consisting of n steps. At each of these steps Xiang takes the two longest pieces a and b, and either bisects $a = a/2 + a/2$ or cuts $a = (a-b) + b$, depending on the specific strategy.

However, for $n = 5$ a counterexample is provided by a (scaled) stick
$$(12.80,\ 6.42,\ 5.35,\ 4.34,\ 2.09)$$
with length 31. If Xiang executes a strategy of the above form, then all $2^5 = 32$ outcomes result in $G > 1$.

¶ Converse direction in general. Although we expect most contestants to solve this part specifically for Liu's division $1:2:4:\dots:2^n$, here we will write the "correct" claim that works in any division.

Claim — Let $a_1, \dots, a_{n+1}$ denote the lengths of Liu's $n+1$ segments. Define

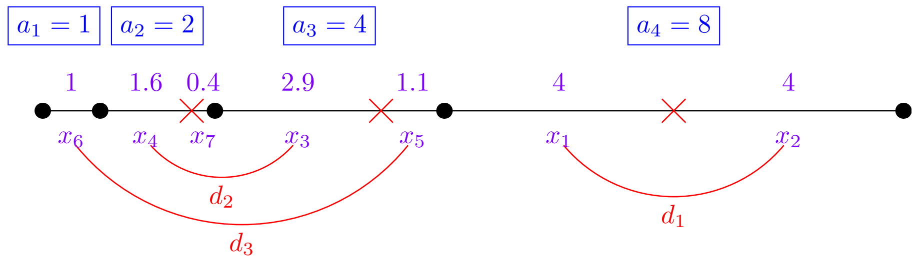
$$\Delta := \min_{\substack{\varepsilon_1,\dots,\varepsilon_{n+1}\in\{-1,0,1\} \\ (\varepsilon_1,\dots,\varepsilon_{n+1})\neq(0,\dots,0)}} \left|\varepsilon_1 a_1 + \varepsilon_2 a_2 + \dots + \varepsilon_{n+1} a_{n+1}\right|$$

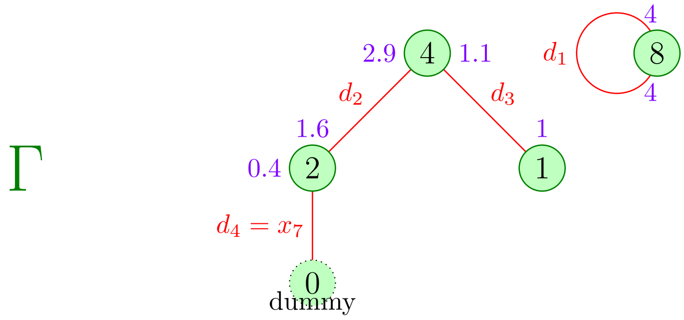
Then regardless of what Xiang does, we always have $G \ge \Delta$.

Proof. Let $d_1 = x_1 - x_2 \ge 0$, $d_2 = x_3 - x_4 \ge 0$, ..., $d_n = x_{2n-1} - x_{2n} \ge 0$. Also let $d_{n+1} := x_{2n+1}$ for convenience, so that

$$G = d_1 + \dots + d_{n+1}.$$

Construct a multigraph Γ on $n+2$ vertices whose vertex set V consists of one vertex for each of the $n+1$ segments, plus a dummy vertex, thought of as having length $a_0 := 0$. For each $1 \le i \le n$, we find which segments x_{2i-1} and x_{2i} were cut from and draw an edge joining them. Finally, we draw one extra edge from the dummy vertex to the segment containing x_{2n+1}. An example when $n=3$ is shown below, scaled by 15. (In the cartoon, each vertex is labeled with the length a_i of its corresponding segment for readability.)





Since Γ has $n+2$ vertices and $n+1$ edges, some component of Γ is a tree (in particular, no parallel edges or self-loops). That tree must be bipartite and we write its vertices as a disjoint union $S \sqcup T$.

Then the main observation is that $|\Sigma(S) - \Sigma(T)|$ (which is among the expressions considered in Δ's definition) is some expression of the form $\pm d_\bullet \pm d_\bullet \pm \dots \pm d_\bullet$, across the edges present in the component, which is certainly at most G. More formally, if $\{d_i\}_{i\in I}$ are the edges in the component, then

$$\Delta \le |\Sigma(S) - \Sigma(T)| = \left|\sum_{i\in I} \pm d_i\right| \le \sum_{i\in I} d_i \le G$$

as desired. $\square$

> **Remark** (Induction)**.** This claim may also be proven by induction on n in a more straightforward way than the trickier argument with the tree above. However, for this induction, stating the fully general claim and using induction on this (rather than just focusing on the $1 : 2 : 4 : \dots : 2^n$ division) is critical.

To complete the proof, just note in the division $1 : 2 : 4 : \dots : 2^n$, the value of Δ indeed equals $\frac{1}{2^{n+1}-1}$ exactly.

> **Remark** (For powers of 2, bipartite coloring is overkill)**.** In the special case where the a_i are powers of 2 — which is all that is needed for the problem — then the graph argument after locating a tree can be simplified. Rather than using a bipartite coloring in the entire tree, one considers the vertex v corresponding to the largest power of 2 in that component, say 2^N. Summing just at that vertex v is already enough to give something of the form
>
> $$2^N = \sum_{d_i \text{ joins } 2^N \text{ to } x_j} (\pm d_i \pm x_j).$$
>
> But the sum of the x_j that appear is at most the sum of the lengths of all the other segments in the component, which in turn is at most $1 + 2 + \dots + 2^{N-1} = 2^N - 1$. This already shows the sum of present d_i must be at least 1.

¶ Converse direction for the specific powers-of-two division. Scaling the stick so it has total length $1 + 2 + \dots + 2^n$, we show a shorter but tricky proof in this specific case that $G \geq 1$. The idea is to prove the following claim by induction on k:

> **Claim** — We always have
>
> $$x_2 + x_4 + \dots + x_{2k} \leq 2^{n-1} + 2^{n-2} + \dots + 2^{n-k}.$$

Proof. Assume not and consider the first k for which this doesn't happen; that means Xiang, on the kth turn, took a piece of length strictly more than 2^{n-k}. This means Xiang's $2k^{\text{th}}$ piece, and thus all $2k$ pieces up until this point in time, came from one of the k segments $\{2^{n-k+1}, 2^{n-k+2}, \dots, 2^n\}$.

However, if all $2k$ pieces come from the k largest segments then

$$x_2 + \dots + x_{2k} \leq \frac{x_1 + x_2 + \dots + x_{2k}}{2} \leq \frac{2^{n-k+1} + 2^{n-k+2} + \dots + 2^n}{2}$$

which produces the required contradiction. □

§2 Solutions to Day 2

§2.1 IMO 2026/4, proposed by SUI

Available online at https://aops.com/community/p38677129.

Problem statement

Shan-Yu and Mulan are playing a game. Let θ be an angle with $0^\circ < \theta < 180^\circ$ known to both players. Initially, Shan-Yu makes a paper triangle $\mathcal{T}$ with measurements of his choice. Then, they repeatedly perform the following steps:

- If $\mathcal{T}$ has at least one angle measuring exactly θ, then the game stops and Mulan wins.
- Otherwise, Mulan chooses a point P on the perimeter of $\mathcal{T}$, different from its three vertices. She then makes a straight cut from P to the opposite vertex of $\mathcal{T}$, splitting it into two triangles.
- Shan-Yu discards one of the two triangles. The remaining triangle becomes the new $\mathcal{T}$.

For which real values of θ can Mulan guarantee her victory in finitely many steps, no matter how Shan-Yu plays?

The answer is that θ should be $\frac{180^\circ}{n}$ for some integer $n \ge 2$.

¶ Proof that $\theta = 180^\circ/n$ can be obtained. We begin with the following observation:

Claim — The game can be won if the triangle ever has an angle of $k\theta$, for any integer $k \ge 1$.

Proof. By induction on $k \ge 1$. In the base case there is nothing to prove. For $k > 1$, split the angle into $\theta + (k-1)\theta$ and apply the induction hypothesis. □

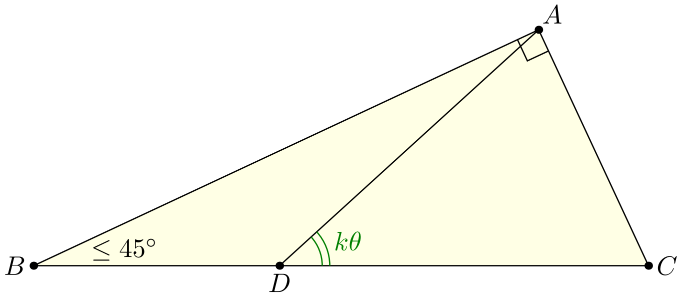
Start by drawing an altitude on the first turn to obtain any right triangle, say ABC with $\angle A = 90^\circ$, and WLOG $\angle B \le 45^\circ$. The game is already won at this point if $n = 2$, so assume henceforth $n \ge 3$. It's not hard to see there exists an integer k such that

$$45^\circ < k\theta \le 90^\circ.$$

Then we can choose D on segment BC with

$$\angle BAD = k\theta - \angle B \iff \angle ADC = k\theta.$$

Hence splitting along line AD completes the solution.

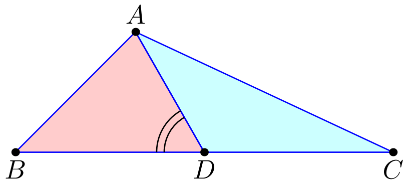


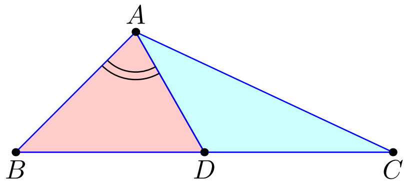
¶ **Proof that $\theta \neq 180°/n$ can be avoided.** We say that an angle is *safe* if it isn't an integer multiple of θ.

It suffices to prove the following result:

Claim — If $\theta \neq \frac{180°}{n}$, then a safe triangle cannot be divided into two unsafe triangles.

Proof. Suppose triangle ABC is safe and divided into two triangles ABD and ACD, with D on side BC. We show that if $\triangle ABD$ is unsafe then $\triangle ACD$ is safe.





Indeed, it suffices to note that

$$\angle CDA = 180° - \angle ADB = \angle B + \angle BAD$$
$$\angle DAC = \angle ADB - \angle C = \angle A - \angle BAD$$

And note the sum/difference of a safe angle and unsafe angle is safe. Hence, if either $\angle ADB$ or $\angle BAD$ are unsafe, both $\angle CDA$ and $\angle DAC$ are safe. $\square$

Hence as long as the initial triangle is safe (which is plainly possible), the second player may ensure all future triangles are safe as well.

§2.2 IMO 2026/5, proposed by Alfrēds Saročinskis (LVA)

Available online at https://aops.com/community/p38677166.

Problem statement

Solve over $\mathbb{R}_{>0}$ the functional equation

$$\sqrt{\frac{x^2+f(y)^2}{2}} \ge \frac{f(x)+y}{2} \ge \sqrt{xf(y)}.$$

The answer is $f(x) \equiv x+c$, for any $c>0$. This works, because

$$\sqrt{\frac{x^2+(y+c)^2}{2}} \ge \underbrace{\frac{(x+c)+y}{2}}_{=\frac{x+(y+c)}{2}} \ge \sqrt{x\cdot(y+c)}$$

is immediate by QM-AM-GM on x and $y+c$.

For the converse direction, we let $P(x,y)$ denote the left inequality and $Q(x,y)$ denote the right inequality. We show two solutions.

¶ Iterated solution. By f^n we mean f applied n times.

Claim — For any $t>0$, the sequence

$$\mathbf{S}(t) := (t, f(t), f^2(t), f^3(t), \dots)$$

forms an arithmetic progression (with nonnegative common difference).

Proof. Note that $P(f(t),t)$ and $Q(f(t),t)$ together gives the equality

$$\frac{f(f(t))+t}{2} = f(t)$$

showing the first three terms are an arithmetic progression. More generally, applying $P(f^n(t), f^{n-1}(t))$ and $Q(f^n(t), f^{n-1}(t))$ for all $n\ge 2$ gives the full result. Since all terms of the sequence are positive, the common difference is nonnegative. □

To finish the problem we just need to show all the common differences that arise this way are the same. The following result proves this except in an annoying edge case of fixed points (which we deal with in the final claim).

Claim — Choose any $x,y>0$, and let A and B be the common differences of $\mathbf{S}(x)$ and $\mathbf{S}(y)$. If $A>0$ and $B>0$, then $A=B$.

Proof. We just prove $A\ge B$ (by symmetry); we only use Q for this. For any positive integers m and n, $Q(f^m(x), f^n(y))$ gives

$$\begin{aligned} &\frac{(x+(m+1)A)+(y+nB)}{2} \ge \sqrt{(x+mA)\cdot(y+(n+1)B)} \\ \iff &\frac{(x+mA)+(y+(n+1)B)}{2} \ge \sqrt{(x+mA)\cdot(y+(n+1)B)} + \frac{B-A}{2} \end{aligned}$$

$$\begin{aligned}&\iff (x+mA)+(y+(n+1)B)-2\sqrt{(x+mA)\cdot(y+(n+1)B)} \ge B-A \\ &\iff \left(\sqrt{x+mA}-\sqrt{y+(n+1)B}\right)^2 \ge B-A.\end{aligned}$$

Now assume for contradiction $0 < A < B$. Then any two terms of the sequences $(\sqrt{x+kA})_{k\ge1}$ and $(\sqrt{y+kB})_{k\ge2}$ need to differ by at least $\varepsilon := \sqrt{B-A} > 0$. However, these sequences are unbounded (as $A, B > 0$) and grow slower than linear (say, eventually consecutive terms differ by less than ε). This produces the required contradiction. $\square$

Assume now that f is not the identity function; then by this point, we know there exists a single constant $D > 0$ such that

$$f(x) \in \{x, x+D\}$$

for every $x > 0$. We need to show in fact $f(x) = x + D$ for all x.

So suppose $f(x) = x + D$ and $f(y) = y$; let's see what we can deduce. In the case $x > y$, we use the estimate

$$P(x,y) \implies \frac{x+y+D}{2} \le \sqrt{\frac{x^2+y^2}{2}} < x \implies x > y + D.$$

Meanwhile, in the case $x < y$ we instead use the estimate

$$Q(y,x) \implies \sqrt{y(x+D)} \le \frac{x+y}{2} < y \implies x < y - D.$$

So combining $P(x,y)$ and $Q(y,x)$ is enough to prove the following statement:

$$f(x) = x + D \text{ and } f(y) = y \implies |x-y| > D. \qquad (\heartsuit)$$

From $(\heartsuit)$ the problem is clearly solved, as its contrapositive says

$$f(x) = x + D \implies f(t) = t + D \ \forall t \in [x-D, x+D]$$

¶ Calculus solution. We briefly outline a calculus-based approach that does not use iterates. From the first solution, we have $f(f(x)) = 2f(x) - x$. Then $P(f(x), y)$ and $Q(f(x), y)$ together read

$$\sqrt{\frac{f(x)^2+f(y)^2}{2}} \ge f(x) + \frac{y-x}{2} \ge \sqrt{f(x)f(y)}.$$

If we isolate $f(y)$ by direct brute-force calculation, the equation above becomes

$$\sqrt{f(x)^2 + 2(y-x)f(x) + \frac{(y-x)^2}{2}} \le f(y) \le f(x) + (y-x) + \frac{(y-x)^2}{4f(x)}.$$

Treating x and hence $f(x)$ as a fixed constant, the above inequality is good enough to imply that:

Claim — For each $x > 0$, we have

$$f(y) = f(x) + (y-x) + O(|y-x|^2) \qquad \text{as} \quad y \to x.$$

Sketch of proof. The upper bound is immediate. The lower bound is slightly more annoying due to the square root; but one may apply Taylor's theorem on the function

$$\sqrt{1+2t+t^2/2} = 1 + t - \frac{t^2}{4} + \frac{t^3}{4} - \frac{9t^4}{32} + \dots$$

near $t=0$; one then selects $t = (y-x)/f(x)$. $\square$

In particular, f is differentiable and its derivative is 1 everywhere; hence $f(x) \equiv x + c$ (and we need $c > 0$).

§2.3 IMO 2026/6, proposed by Hung-Hsun Hans Yu (TWN)

Available online at https://aops.com/community/p38677077.

Problem statement

Let a_1, a_2, a_3, ... be an infinite sequence of positive integers greater than 1. Suppose that for all positive integers n, the number a_{n+1} is the smallest positive integer greater than a_n such that $\gcd(a_{n+1}, a_i) > 1$ for every $i = 1, 2, \dots, n$. Prove that there exist positive integers T and L such that

$$a_{n+T} = a_n + L$$

for every positive integer n.

The basic idea is that there should only be finitely many primes that "matter".

Remark (Example with $a_1 = 15$)**.** It will be easier to follow the proof below from the following example. If $a_1 = 15$, then the terms are

$$(a_n)_{n\geq 1} = (15, 18, 20, 24, 30, 36, 40, 42, 45, 48, 50, 54, \dots)$$

and one can give the following description of a_n: it consists of exactly the set of positive integers starting from 15 divisible by at least one of $\{2 \cdot 3, 3 \cdot 5, 2 \cdot 5\}$. As an example, the factor of 7 in 42 is irrelevant; 18 does all the work.

In particular, membership in the sequence only depends on $a_n \pmod{30}$. One can thus take $L = 30$ and T the number of residues modulo 30 that appear (in this case, $T = 8$).

To make things precise, define a partial order on the sequences where we define the relation $\prec$ on the a_i's by

$$a_m \prec a_n \iff m < n \text{ and } \operatorname{rad} a_m \mid \operatorname{rad} a_n.$$

(For example, when $a_1 = 15$ we have $18 \prec 42$.) The point is that if $a_m \prec a_n$ then a_n is irrelevant. To be explicit, we state the following (trivial) claim:

Claim — The following conditions are equivalent for an integer $x \geq 1$:

- x appears in the sequence;
- $\gcd(x, a_i) > 1$ for every $i \geq 1$;
- $\gcd(x, a_i) > 1$ for every $i \geq 1$ such that $a_i < x$;
- $\gcd(x, a_i) > 1$ for every $i \geq 1$ such that a_i is $\prec$-minimal;
- $\gcd(x, a_i) > 1$ for every $i \geq 1$ such that a_i is $\prec$-minimal and $a_i < x$.

Proof. All of these follow directly from the definition. □

We'll now say a prime is *large* if it is greater than a_1^2 and *small* otherwise. The main heart of the proof is:

Claim — If a_n is divisible by a large prime, then it is not $\prec$-minimal.

Proof. By induction on $n \geq 1$; assume $a_n = pc$ where p is large. Let q be any prime divisor of $\gcd(a_1, a_n)$. Consider the numbers

$$\{c,\ qc,\ q^2c,\ q^3c,\ \dots\}.$$

At least one of them must lie in the half-open interval $[a_1, a_n)$, since

$$\frac{a_n}{a_1} \geq \frac{p}{a_1} > a_1 \geq q.$$

We contend $q^k c$ must exist in the sequence. Indeed, for every a_i which is $\prec$-minimal and $a_i < a_n$, we get $p \nmid a_i$ (by induction hypothesis) and thus $\gcd(q^k c, a_i) \geq \gcd(a_n, a_i) > 1$. Hence $q^k c$ appears in the sequence, showing a_n is not $\prec$-minimal. $\square$

The rest of the proof is bookkeeping. Let P be the set of small primes (at most a_1^2), and for each $a \in \mathbb{N}$ let

$$S(a) := \{\text{small primes } p \text{ dividing } a\} \subseteq P, \qquad \mathcal{F} := \{S(a_n) \mid n \geq 1\} \subseteq 2^P.$$

Then the sequence can be described exactly as follows: it consists of all integers $a \geq a_1$ with $S(a) \in \mathcal{F}$. In particular, membership in our sequence depends only on

$$a \bmod \prod_{p \in P} p.$$

So we can choose $L = \prod_{p \in P} p$ and T as the number of residues that appear. The proof is complete.

Remark. With a little bit of additional care one can show in fact that when a_n is $\prec$-minimal, its prime divisors are actually at most a_1 (rather than just less than a_1^2).

Remark. Some property of the ordering of the integers is needed to solve the problem. Indeed, some coordinators tried to prove the following totally combinatorial statement: Suppose $\mathcal{F}$ is a family of finite sets of prime numbers such that

$$S \in \mathcal{F} \iff (S \cap T \neq \varnothing \text{ for all } T \in \mathcal{F})$$

holds for all finite sets S of prime numbers. Then the number of minimal sets in $\mathcal{F}$ (by inclusion) is finite. (For example, if $a_1 = 15$ the analogous $\mathcal{F}$ has three minimal sets, $\{2,3\}$, $\{2,5\}$, and $\{3,5\}$.)

However, this combinatorial version of the problem is false. Colloquially, if one pick as different order of the positive integers in the problem statement, the main claim about large primes may no longer hold.