Team Selection Test for the 66th International Mathematical Olympiad and 14th European Girls Math Olympiad

United States of America

Day I

Thursday, December 12, 2024

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

You may keep the problems, but you cannot discuss them publicly until they are posted by staff online.

Problem 1. Let n be a positive integer. Ana and Banana play a game. Banana thinks of a function $f\colon \mathbb{Z} \to \mathbb{Z}$ and a prime number p. He tells Ana that f is nonconstant, $p < 100$, and $f(x+p) = f(x)$ for all integers x. Ana's goal is to determine the value of p. She writes down n integers $x_1, \ldots, x_n$. After seeing this list, Banana writes down $f(x_1), \ldots, f(x_n)$ in order. Ana wins if she can determine the value of p from this information. Find the smallest value of n for which Ana has a winning strategy.

Problem 2. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of real numbers for which $a_1 > b_1$ and

$$a_{n+1} = a_n^2 - 2b_n$$
$$b_{n+1} = b_n^2 - 2a_n$$

for all positive integers n. Prove that $a_1, a_2, \ldots$ is eventually increasing (that is, there exists a positive integer N for which $a_k < a_{k+1}$ for all $k > N$).

Problem 3. Let $A_1A_2\cdots A_{2025}$ be a convex 2025-gon, and let $A_i = A_{i+2025}$ for all integers i. Distinct points P and Q lie in its interior such that $\angle A_{i-1}A_iP = \angle QA_iA_{i+1}$ for all i. Define points P_i^j and Q_i^j for integers i and positive integers j as follows:

- For all i, $P_i^1 = Q_i^1 = A_i$.
- For all i and j, P_i^{j+1} and Q_i^{j+1} are the circumcenters of $PP_i^jP_{i+1}^j$ and $QQ_i^jQ_{i+1}^j$, respectively.

Let $\mathcal{P}$ and $\mathcal{Q}$ be the polygons $P_1^{2025}P_2^{2025}\cdots P_{2025}^{2025}$ and $Q_1^{2025}Q_2^{2025}\cdots Q_{2025}^{2025}$, respectively.

(a) Prove that $\mathcal{P}$ and $\mathcal{Q}$ are cyclic.

(b) Let O_P and O_Q be the circumcenters of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Assuming that $O_P \neq O_Q$, show that O_PO_Q is parallel to PQ.

Team Selection Test for the 66th International Mathematical Olympiad and 14th European Girls Math Olympiad

United States of America

Day II

Thursday, January 9, 2024

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

You may keep the problems, but you cannot discuss them publicly until they are posted by staff online.

Problem 4. Let ABC be a triangle, and let X, Y, and Z be collinear points such that $AY = AZ$, $BZ = BX$, and $CX = CY$. Points X', Y', and Z' are the reflections of X, Y, and Z over BC, CA, and AB, respectively. Prove that if $X'Y'Z'$ is a nondegenerate triangle, then its circumcenter lies on the circumcircle of ABC.

Problem 5. A pond has 2025 lily pads arranged in a circle. Two frogs, Alice and Bob, begin on different lily pads. A frog jump is a jump which travels 2, 3, or 5 positions clockwise. Alice and Bob each make a series of frog jumps, and each frog ends on the same lily pad that it started from. Given that each lily pad is the destination of exactly one jump, prove that each frog completes exactly two laps around the pond (i.e. travels 4050 positions in total).

Problem 6. Prove that there exists a real number $\varepsilon > 0$ such that there are infinitely many sequences of integers $0 < a_1 < a_2 < \ldots < a_{2025}$ satisfying

$$\gcd(a_1^2 + 1, a_2^2 + 1, \ldots, a_{2025}^2 + 1) > a_{2025}^{1+\varepsilon}.$$