Team Selection Test for the 65^{th} International Mathematical Olympiad and 13^{th} European Girls Math Olympiad

United States of America

Day I

Thursday, December 7, 2023

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, December 11 at noon Eastern time.

TST 1. Find the smallest constant $C > 1$ such that the following statement holds: for every integer $n \geq 2$ and sequence of non-integer positive real numbers $a_1, a_2, \ldots, a_n$ satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1,$$

it's possible to choose positive integers b_i such that

(i) for each $i = 1, 2, \ldots, n$, either $b_i = \lfloor a_i \rfloor$ or $b_i = \lfloor a_i \rfloor + 1$; and

(ii) we have

$$1 < \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \leq C.$$

(Here $\lfloor \bullet \rfloor$ denotes the floor function, as usual.)

TST 2. Let ABC be a triangle with incenter I. Let segment AI intersect the incircle of triangle ABC at point D. Suppose that line BD is perpendicular to line AC. Let P be a point such that $\angle BPA = \angle PAI = 90^\circ$. Point Q lies on segment BD such that the circumcircle of triangle ABQ is tangent to line BI. Point X lies on line PQ such that $\angle IAX = \angle XAC$. Prove that $\angle AXP = 45^\circ$.

TST 3. Let $n > k \geq 1$ be integers and let p be a prime dividing $\binom{n}{k}$. Prove that the k-element subsets of $\{1, \ldots, n\}$ can be split into p classes of equal size, such that any two subsets with the same sum of elements belong to the same class.

Team Selection Test for the 65$^{\text{th}}$ International Mathematical Olympiad and 13$^{\text{th}}$ European Girls Math Olympiad

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Day II

Thursday, January 11, 2024

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, January 15 at noon Eastern time.

TST 4. Find all integers $n \geq 2$ for which there exists a sequence of $2n$ pairwise distinct points $(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$ in the plane satisfying the following four conditions:

(i) no three of the $2n$ points are collinear;

(ii) $P_iP_{i+1} \geq 1$ for all $i = 1, 2, \ldots, n$, where $P_{n+1} = P_1$;

(iii) $Q_iQ_{i+1} \geq 1$ for all $i = 1, 2, \ldots, n$, where $Q_{n+1} = Q_1$; and

(iv) $P_iQ_j \leq 1$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

TST 5. Suppose $a_1 < a_2 < \cdots < a_{2024}$ is an arithmetic sequence of positive integers, and $b_1 < b_2 < \cdots < b_{2024}$ is a geometric sequence of positive integers. Find the maximum possible number of integers that could appear in both sequences, over all possible choices of the two sequences.

TST 6. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$f(xf(y)) + f(y) = f(x + y) + f(xy).$$