

# IMO 2023 Solution Notes

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This is a compilation of solutions for the 2023 IMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the “official” solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered “standard”, then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like “let  $\mathbb{R}$  denote the set of real numbers” are typically omitted entirely.

Corrections and comments are welcome!

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## §0 Problems

1. Determine all composite integers  $n > 1$  that satisfy the following property: if  $d_1 < d_2 < \dots < d_k$  are all the positive divisors of  $n$  with then  $d_i$  divides  $d_{i+1} + d_{i+2}$  for every  $1 \leq i \leq k - 2$ .
2. Let  $ABC$  be an acute-angled triangle with  $AB < AC$ . Let  $\Omega$  be the circumcircle of  $ABC$ . Let  $S$  be the midpoint of the arc  $CB$  of  $\Omega$  containing  $A$ . The perpendicular from  $A$  to  $BC$  meets  $BS$  at  $D$  and meets  $\Omega$  again at  $E \neq A$ . The line through  $D$  parallel to  $BC$  meets line  $BE$  at  $L$ . Denote the circumcircle of triangle  $BDL$  by  $\omega$ . Let  $\omega$  meet  $\Omega$  again at  $P \neq B$ . Prove that the line tangent to  $\omega$  at  $P$  meets line  $BS$  on the internal angle bisector of  $\angle BAC$ .
3. For each integer  $k \geq 2$ , determine all infinite sequences of positive integers  $a_1, a_2, \dots$  for which there exists a polynomial  $P$  of the form

$$P(x) = x^k + c_{k-1}x^{k-1} + \dots + c_1x + c_0,$$

where  $c_0, c_1, \dots, c_{k-1}$  are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2} \cdots a_{n+k}$$

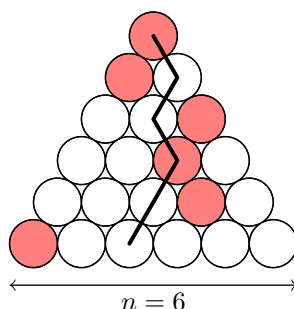
for every integer  $n \geq 1$ .

4. Let  $x_1, x_2, \dots, x_{2023}$  be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \dots + x_n) \left( \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right)}$$

is an integer for every  $n = 1, 2, \dots, 2023$ . Prove that  $a_{2023} \geq 3034$ .

5. Let  $n$  be a positive integer. A *Japanese triangle* consists of  $1 + 2 + \dots + n$  circles arranged in an equilateral triangular shape such that for each  $1 \leq i \leq n$ , the  $i^{\text{th}}$  row contains exactly  $i$  circles, exactly one of which is colored red. A *ninja path* in a Japanese triangle is a sequence of  $n$  circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with  $n = 6$ , along with a ninja path in that triangle containing two red circles.



In terms of  $n$ , find the greatest  $k$  such that in each Japanese triangle there is a ninja path containing at least  $k$  red circles.

6. Let  $ABC$  be an equilateral triangle. Let  $A_1, B_1, C_1$  be interior points of  $ABC$  such that  $BA_1 = A_1C$ ,  $CB_1 = B_1A$ ,  $AC_1 = C_1B$ , and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ.$$

Let  $A_2 = \overline{BC_1} \cap \overline{CB_1}$ ,  $B_2 = \overline{CA_1} \cap \overline{AC_1}$ ,  $C_2 = \overline{AB_1} \cap \overline{BA_1}$ . Prove that if triangle  $A_1B_1C_1$  is scalene, then the circumcircles of triangles  $AA_1A_2$ ,  $BB_1B_2$ , and  $CC_1C_2$  all pass through two common points.

## §1 Solutions to Day 1

### §1.1 IMO 2023/1, proposed by Santiago Rodriguez (COL)

Available online at <https://aops.com/community/p28097575>.

#### Problem statement

Determine all composite integers  $n > 1$  that satisfy the following property: if  $d_1 < d_2 < \dots < d_k$  are all the positive divisors of  $n$  with then  $d_i$  divides  $d_{i+1} + d_{i+2}$  for every  $1 \leq i \leq k - 2$ .

The answer is prime powers.

¶ **Verification that these work.** When  $n = p^e$ , we get  $d_i = p^{i-1}$ . The  $i^{\text{th}}$  relationship reads

$$p^{i-1} \mid p^i + p^{i+1}$$

which is obviously true.

¶ **Proof that these are the only answers.** Conversely, suppose  $n$  has at least two distinct prime divisors. Let  $p < q$  denote the two smallest ones, and let  $p^e$  be the largest power of  $p$  which both divides  $n$  and is less than  $q$ , hence  $e \geq 1$ . Then the smallest factors of  $n$  are  $1, p, \dots, p^e, q$ . So we are supposed to have

$$\frac{n}{q} \mid \frac{n}{p^e} + \frac{n}{p^{e-1}} = \frac{(p+1)n}{p^e}$$

which means that the ratio

$$\frac{q(p+1)}{p^e}$$

needs to be an integer, which is obviously not possible.

## §1.2 IMO 2023/2, proposed by Tiago Mourão and Nuno Arala (POR)

Available online at <https://aops.com/community/p28097552>.

### Problem statement

Let  $ABC$  be an acute-angled triangle with  $AB < AC$ . Let  $\Omega$  be the circumcircle of  $ABC$ . Let  $S$  be the midpoint of the arc  $CB$  of  $\Omega$  containing  $A$ . The perpendicular from  $A$  to  $BC$  meets  $BS$  at  $D$  and meets  $\Omega$  again at  $E \neq A$ . The line through  $D$  parallel to  $BC$  meets line  $BE$  at  $L$ . Denote the circumcircle of triangle  $BDL$  by  $\omega$ . Let  $\omega$  meet  $\Omega$  again at  $P \neq B$ . Prove that the line tangent to  $\omega$  at  $P$  meets line  $BS$  on the internal angle bisector of  $\angle BAC$ .

**Claim —** We have  $LPS$  collinear.

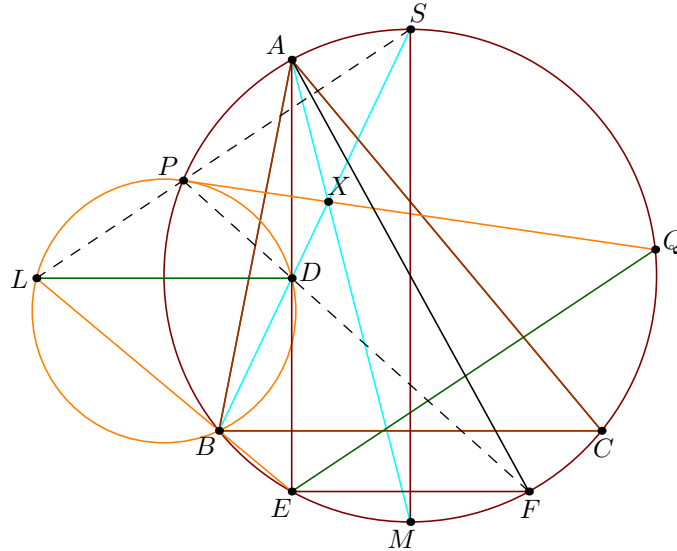
*Proof.* Because  $\angle LPB = \angle LDB = \angle CBD = \angle CBS = \angle SCB = \angle SPB$ .  $\square$

Let  $F$  be the antipode of  $A$ , so  $AMFS$  is a rectangle.

**Claim —** We have  $PDF$  collinear. (This lets us erase  $L$ .)

*Proof.* Because  $\angle SPD = \angle LPD = \angle LBD = \angle SBE = \angle FCS = \angle FPS$ .  $\square$

Let us define  $X = \overline{AM} \cap \overline{BS}$  and complete chord  $\overline{PXQ}$ . We aim to show that  $\overline{PXQ}$  is tangent to  $(PDLB)$ .



**Claim (Main projective claim) —** We have  $XP = XA$ .

*Proof.* Introduce  $Y = \overline{PDF} \cap \overline{AM}$ . Note that

$$-1 = (SM; EF) \stackrel{A}{=} (S, X; D, \overline{AF} \cap \overline{ES}) \stackrel{F}{=} (\infty X; YA)$$

where  $\infty = \overline{AM} \cap \overline{SF}$  is at infinity (because  $AMSF$  is a rectangle). Thus,  $XY = XA$ .



### §1.3 IMO 2023/3, proposed by Ivan Chan (MAS)

Available online at <https://aops.com/community/p28097600>.

#### Problem statement

For each integer  $k \geq 2$ , determine all infinite sequences of positive integers  $a_1, a_2, \dots$  for which there exists a polynomial  $P$  of the form

$$P(x) = x^k + c_{k-1}x^{k-1} + \dots + c_1x + c_0,$$

where  $c_0, c_1, \dots, c_{k-1}$  are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2} \dots a_{n+k}$$

for every integer  $n \geq 1$ .

The answer is  $a_n$  being an arithmetic progression. Indeed, if  $a_n = d(n-1) + a_1$  for  $d \geq 0$  and  $n \geq 1$ , then

$$a_{n+1}a_{n+2} \dots a_{n+k} = (a_n + d)(a_n + 2d) \dots (a_n + kd)$$

so we can just take  $P(x) = (x+d)(x+2d) \dots (x+kd)$ .

The converse direction takes a few parts.

**Claim —** Either  $a_1 < a_2 < \dots$  or the sequence is constant.

*Proof.* Note that

$$\begin{aligned} P(a_{n-1}) &= a_n a_{n+1} \dots a_{n+k-1} \\ P(a_n) &= a_{n+1} a_{n+2} \dots a_{n+k} \\ \implies a_{n+k} &= \frac{P(a_n)}{P(a_{n-1})} \cdot a_n. \end{aligned}$$

Now the polynomial  $P$  is strictly increasing over  $\mathbb{N}$ .

So assume for contradiction there's an index  $n$  such that  $a_n < a_{n-1}$ . Then in fact the above equation shows  $a_{n+k} < a_n < a_{n-1}$ . Then there's an index  $\ell \in [n+1, n+k]$  such that  $a_\ell < a_{\ell-1}$ , and also  $a_\ell < a_n$ . Continuing in this way, we can find an infinite descending subsequence of  $(a_n)$ , but that's impossible because we assumed integers.

Hence we have  $a_1 \leq a_2 \leq \dots$ . Now similarly, if  $a_n = a_{n-1}$  for any index  $n$ , then  $a_{n+k} = a_n$ , ergo  $a_{n-1} = a_n = a_{n+1} = \dots = a_{n+k}$ . So the sequence is eventually constant, and then by downwards induction, it is fully constant.  $\square$

**Claim —** There exists a constant  $C$  (depending only  $P, k$ ) such that we have  $a_{n+1} \leq a_n + C$ .

*Proof.* Let  $C$  be a constant such that  $P(x) < x^k + Cx^{k-1}$  for all  $x \in \mathbb{N}$  (for example  $C = c_0 + c_1 + \dots + c_{k-1} + 1$  works). We have

$$a_{n+k} = \frac{P(a_n)}{a_{n+1}a_{n+2} \dots a_{n+k-1}}$$

$$\begin{aligned}
&< \frac{P(a_n)}{(a_n + 1)(a_n + 2) \dots (a_n + k - 1)} \\
&< \frac{a_n^k + C \cdot a_n^{k-1}}{(a_n + 1)(a_n + 2) \dots (a_n + k - 1)} \\
&< a_n + C + 1.
\end{aligned}
\quad \square$$

Assume henceforth  $a_n$  is nonconstant, and hence unbounded. For each index  $n$  and term  $a_n$  in the sequence, consider the associated differences  $d_1 = a_{n+1} - a_n$ ,  $d_2 = a_{n+2} - a_{n+1}$ ,  $\dots$ ,  $d_k = a_{n+k} - a_{n+k-1}$ , which we denote by

$$\Delta(n) := (d_1, \dots, d_k).$$

This  $\Delta$  can only take up to  $C^k$  different values. So in particular, some tuple  $(d_1, \dots, d_k)$  must appear infinitely often as  $\Delta(n)$ ; for that tuple, we obtain

$$P(a_N) = (a_N + d_1)(a_N + d_1 + d_2) \dots (a_N + d_1 + \dots + d_k)$$

for infinitely many  $N$ . But because of that, we actually must have

$$P(X) = (X + d_1)(X + d_1 + d_2) \dots (X + d_1 + \dots + d_k).$$

However, this *also* means that *exactly* one output to  $\Delta$  occurs infinitely often (because that output is determined by  $P$ ). Consequently, it follows that  $\Delta$  is eventually constant. For this to happen,  $a_n$  must eventually coincide with an arithmetic progression of some common difference  $d$ , and  $P(X) = (X + d)(X + 2d) \dots (X + kd)$ . Finally, this implies by downwards induction that  $a_n$  is an arithmetic progression on all inputs.

## §2 Solutions to Day 2

### §2.1 IMO 2023/4, proposed by Merlijn Staps (NLD)

Available online at <https://aops.com/community/p28104298>.

#### Problem statement

Let  $x_1, x_2, \dots, x_{2023}$  be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \dots + x_n) \left( \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right)}$$

is an integer for every  $n = 1, 2, \dots, 2023$ . Prove that  $a_{2023} \geq 3034$ .

Note that  $a_{n+1} > \sqrt{\sum_1^n x_i \sum_1^n \frac{1}{x_i}} = a_n$  for all  $n$ , so that  $a_{n+1} \geq a_n + 1$ . Observe  $a_1 = 1$ . We are going to prove that

$$a_{2m+1} \geq 3m + 1 \quad \text{for all } m \geq 0$$

by induction on  $m$ , with the base case being clear.

We now present two variations of the induction. The first shorter solution compares  $a_{n+2}$  directly to  $a_n$ , showing it increases by at least 3. Then we give a longer approach that compares  $a_{n+1}$  to  $a_n$ , and shows it cannot increase by 1 twice in a row.

¶ **Induct-by-two solution.** Let  $u = \sqrt{\frac{x_{n+1}}{x_{n+2}}} \neq 1$ . Note that by using Cauchy-Schwarz with three terms:

$$\begin{aligned} a_{n+2}^2 &= \left[ (x_1 + \dots + x_n) + x_{n+1} + x_{n+2} \right] \left[ \left( \frac{1}{x_1} + \dots + \frac{1}{x_n} \right) + \frac{1}{x_{n+2}} + \frac{1}{x_{n+1}} \right] \\ &\geq \left( \sqrt{(x_1 + \dots + x_n) \left( \frac{1}{x_1} + \dots + \frac{1}{x_n} \right)} + \sqrt{\frac{x_{n+1}}{x_{n+2}}} + \sqrt{\frac{x_{n+2}}{x_{n+1}}} \right)^2 \\ &= \left( a_n + u + \frac{1}{u} \right)^2 \\ \implies a_{n+2} &\geq a_n + u + \frac{1}{u} > a_n + 2 \end{aligned}$$

where the last equality  $u + \frac{1}{u} > 2$  is by AM-GM, strict as  $u \neq 1$ . It follows that  $a_{n+2} \geq a_n + 3$ , completing the proof.

¶ **Induct-by-one solution.** The main claim is:

**Claim —** It's impossible to have  $a_n = c$ ,  $a_{n+1} = c + 1$ ,  $a_{n+2} = c + 2$  for any  $c$  and  $n$ .

*Proof.* Let  $p = x_{n+1}$  and  $q = x_{n+2}$  for brevity. Let  $s = \sum_1^n x_i$  and  $t = \sum_1^n \frac{1}{x_i}$ , so  $c^2 = a_n^2 = st$ .

From  $a_n = c$  and  $a_{n+1} = c + 1$  we have

$$(c + 1)^2 = a_{n+1}^2 = (p + s) \left( \frac{1}{p} + t \right)$$

$$\begin{aligned} &= st + pt + \frac{1}{p}s + 1 = c^2 + pt + \frac{1}{p}s + 1 \\ &\stackrel{\text{AM-GM}}{\geq} c^2 + 2\sqrt{st} + 1 = c^2 + 2\sqrt{c^2} + 1 = (c + 1)^2. \end{aligned}$$

Hence, equality must hold in the AM-GM we must have exactly

$$pt = \frac{1}{p}s = c.$$

If we repeat the argument again on  $a_{n+1} = c + 1$  and  $a_{n+2} = c + 2$ , then

$$q \left( \frac{1}{p} + t \right) = \frac{1}{q} (p + s) = c + 1.$$

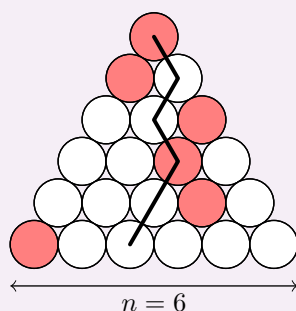
However this forces  $\frac{p}{q} = \frac{q}{p} = 1$  which is impossible. □

## §2.2 IMO 2023/5, proposed by Merlijn Staps and Daniël Kroes (NLD)

Available online at <https://aops.com/community/p28104367>.

### Problem statement

Let  $n$  be a positive integer. A *Japanese triangle* consists of  $1 + 2 + \cdots + n$  circles arranged in an equilateral triangular shape such that for each  $1 \leq i \leq n$ , the  $i^{\text{th}}$  row contains exactly  $i$  circles, exactly one of which is colored red. A *ninja path* in a Japanese triangle is a sequence of  $n$  circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with  $n = 6$ , along with a ninja path in that triangle containing two red circles.

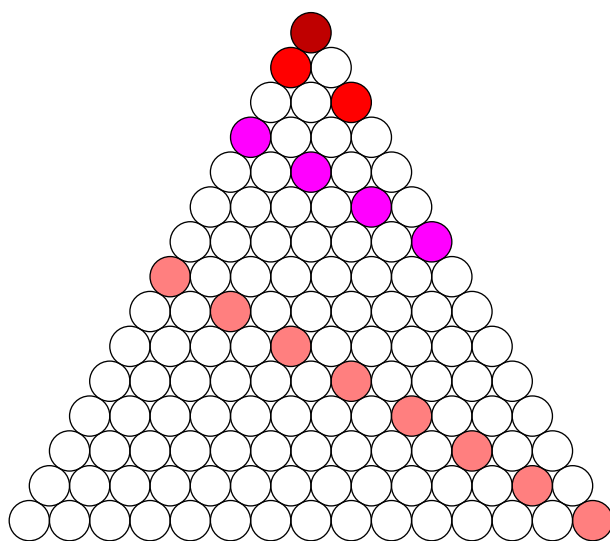


In terms of  $n$ , find the greatest  $k$  such that in each Japanese triangle there is a ninja path containing at least  $k$  red circles.

The answer is

$$k = \lfloor \log_2(n) \rfloor + 1.$$

¶ **Construction.** It suffices to find a Japanese triangle for  $n = 2^e - 1$  with the property that at most  $e$  red circles in any ninja path. The construction shown below for  $e = 4$  obviously generalizes, and works because in each of the sets  $\{1\}$ ,  $\{2, 3\}$ ,  $\{4, 5, 6, 7\}$ ,  $\dots$ ,  $\{2^{e-1}, \dots, 2^e - 1\}$ , at most one red circle can be taken. (These sets are colored in different shades of red for visual clarity).



¶ **Bound.** Conversely, we show that in any Japanese triangle, one can find a ninja path containing at least

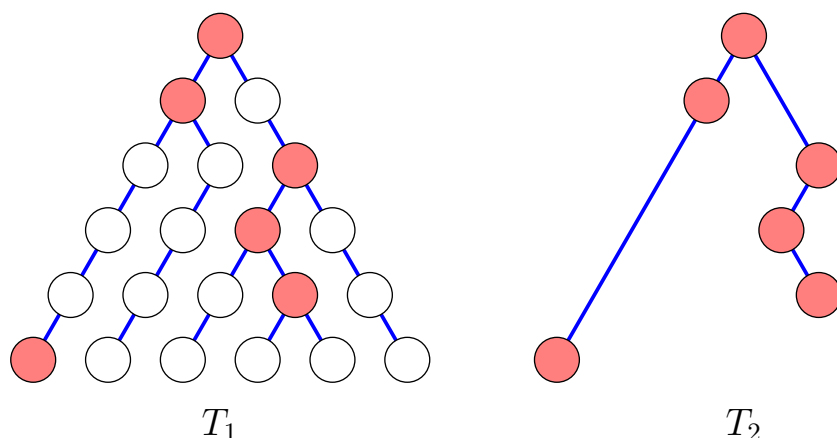
$$k = \lfloor \log_2(n) \rfloor + 1.$$

The following short solution was posted at <https://aops.com/community/p28134004>, apparently first found by the team leader for Iran.

We construct a rooted binary tree  $T_1$  on the set of all circles as follows. For each row, other than the bottom row:

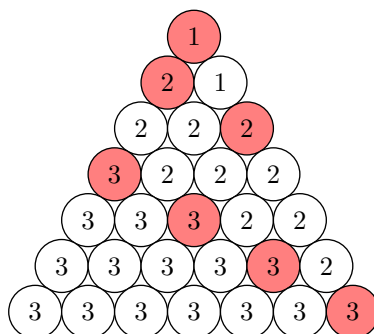
- Connect the red circle to both circles under it;
- White circles to the left of the red circle in its row are connected to the left;
- White circles to the right of the red circle in its row are connected to the right.

The circles in the bottom row are all leaves of this tree. For example, the  $n = 6$  construction in the beginning gives the tree shown on the left half of the figure below:



Now focus on only the red circles, as shown in the right half of the figure. We build a new rooted tree  $T_2$  where each red circle is joined to the red circle below it if there was a path of (zero or more) white circles in  $T_1$  between them. Then each red circle has at most 2 direct descendants in  $T_2$ . Hence the depth of the new tree  $T_2$  exceeds  $\log_2(n)$ , which produces the desired path.

¶ **Another recursive proof of bound, communicated by Helio Ng.** We give another proof that  $\lfloor \log_2 n \rfloor + 1$  is always achievable. Define  $f(i, j)$  to be the maximum number of red circles contained in the portion of a ninja path from  $(1, 1)$  to  $(i, j)$ , including the endpoints  $(1, 1)$  and  $(i, j)$ . (If  $(i, j)$  is not a valid circle in the triangle, define  $f(i, j) = 0$  for convenience.) An example is shown below with the values of  $f(i, j)$  drawn in the circles.



We have that

$$f(i, j) = \max \{f(i-1, j-1), f(i, j-1)\} + \begin{cases} 1 & \text{if } (i, j) \text{ is red} \\ 0 & \text{otherwise} \end{cases}$$

since every ninja path passing through  $(i, j)$  also passes through either  $(i-1, j-1)$  or  $(i, j-1)$ . Now consider the quantity  $S_j = f(0, j) + \dots + f(j, j)$ . We obtain the following recurrence:

**Claim —**  $S_{j+1} \geq S_j + \left\lceil \frac{S_j}{j} \right\rceil + 1.$

*Proof.* Consider a maximal element  $f(m, j)$  of  $\{f(0, j), \dots, f(j, j)\}$ . We perform the following manipulations:

$$\begin{aligned} S_{j+1} &= \sum_{i=0}^{j+1} \max \{f(i-1, j), f(i, j)\} + \sum_{i=0}^{j+1} \begin{cases} 1 & \text{if } (i, j+1) \text{ is red} \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{i=0}^m \max \{f(i-1, j), f(i, j)\} + \sum_{i=m+1}^j \max \{f(i-1, j), f(i, j)\} + 1 \\ &\geq \sum_{i=0}^m f(i, j) + \sum_{i=m+1}^j f(i-1, j) + 1 \\ &= S_j + f(m, j) + 1 \\ &\geq S_j + \left\lceil \frac{S_j}{j} \right\rceil + 1 \end{aligned}$$

where the last inequality is due to Pigeonhole.  $\square$

This is actually enough to solve the problem. Write  $n = 2^c + r$ , where  $0 \leq r \leq 2^c - 1$ .

**Claim —**  $S_n \geq cn + 2r + 1$ . In particular,  $\left\lceil \frac{S_n}{n} \right\rceil \geq c + 1$ .

*Proof.* First note that  $S_n \geq cn + 2r + 1$  implies  $\left\lceil \frac{S_n}{n} \right\rceil \geq c + 1$  because

$$\left\lceil \frac{S_n}{n} \right\rceil \geq \left\lceil \frac{cn + 2r + 1}{n} \right\rceil = c + \left\lceil \frac{2r + 1}{n} \right\rceil = c + 1.$$

We proceed by induction on  $n$ . The base case  $n = 1$  is clearly true as  $S_1 = 1$ . Assuming that the claim holds for some  $n = j$ , we have

$$\begin{aligned} S_{j+1} &\geq S_j + \left\lceil \frac{S_j}{j} \right\rceil + 1 \\ &\geq cj + 2r + 1 + c + 1 + 1 \\ &= c(j+1) + 2(r+1) + 1 \end{aligned}$$

so the claim is proved for  $n = j + 1$  if  $j + 1$  is not a power of 2. If  $j + 1 = 2^{c+1}$ , then by writing  $c(j+1) + 2(r+1) + 1 = c(j+1) + (j+1) + 1 = (c+2)(j+1) + 1$ , the claim is also proved.  $\square$

Now  $\left\lceil \frac{S_n}{n} \right\rceil \geq c + 1$  implies the existence of some ninja path containing at least  $c + 1$  red circles, and we are done.

**§2.3 IMO 2023/6, proposed by Ankan Bhattacharya, Luke Robitaille (USA)**

Available online at <https://aops.com/community/p28104331>.

**Problem statement**

Let  $ABC$  be an equilateral triangle. Let  $A_1, B_1, C_1$  be interior points of  $ABC$  such that  $BA_1 = A_1C$ ,  $CB_1 = B_1A$ ,  $AC_1 = C_1B$ , and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ.$$

Let  $A_2 = \overline{BC_1} \cap \overline{CB_1}$ ,  $B_2 = \overline{CA_1} \cap \overline{AC_1}$ ,  $C_2 = \overline{AB_1} \cap \overline{BA_1}$ . Prove that if triangle  $A_1B_1C_1$  is scalene, then the circumcircles of triangles  $AA_1A_2$ ,  $BB_1B_2$ , and  $CC_1C_2$  all pass through two common points.

This is the second official solution from the marking scheme, also communicated to me by Michael Ren. Define  $O$  as the center of  $ABC$  and set the angles

$$\alpha := \angle A_1CB = \angle CBA_1$$

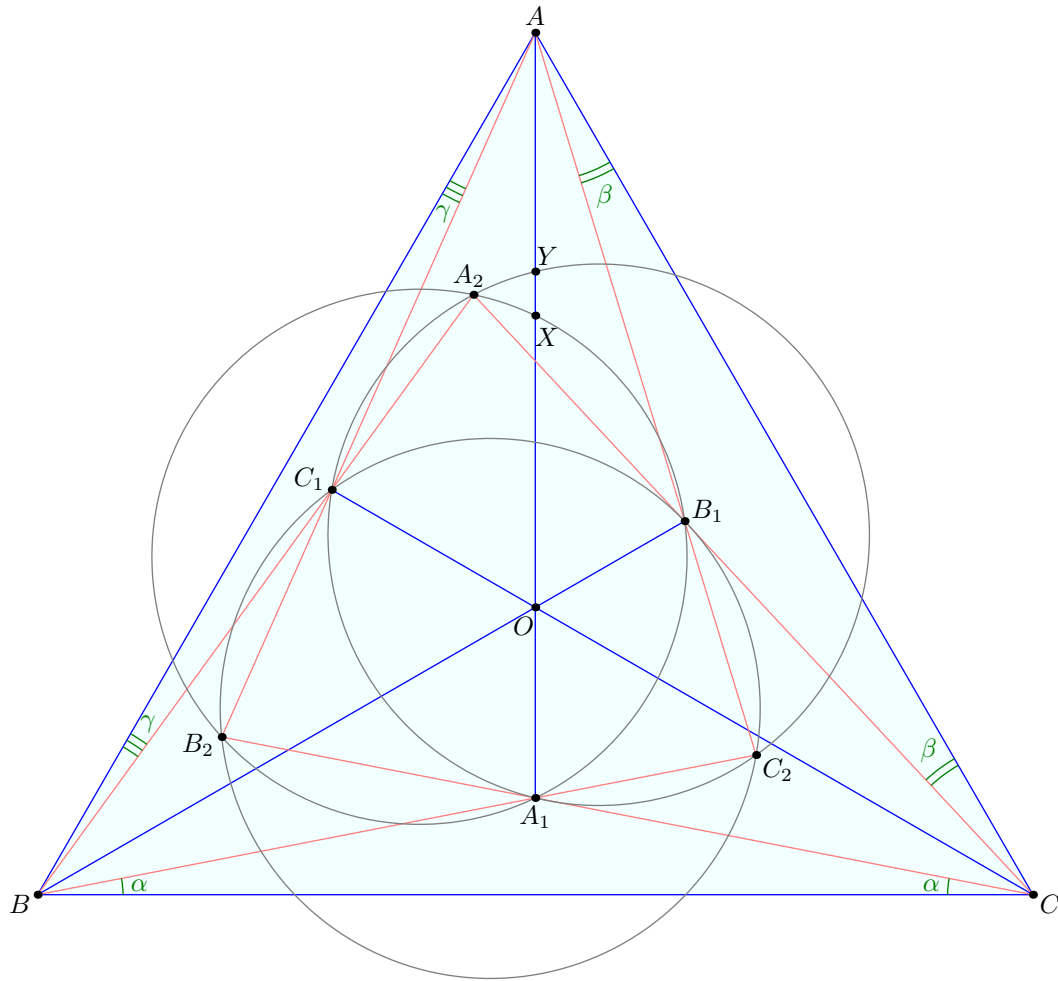
$$\beta := \angle ACB_1 = \angle B_1AC$$

$$\gamma := \angle C_1AB = \angle C_1BA$$

so that

$$\alpha + \beta + \gamma = 30^\circ.$$

In particular,  $\max(\alpha, \beta, \gamma) < 30^\circ$ , so it follows that  $A_1$  lies inside  $\triangle OBC$ , and similarly for the others. This means for example that  $C_1$  lies between  $B$  and  $A_2$ , and so on. Therefore the polygon  $A_2C_1B_2A_1C_2B_1$  is convex.



We start by providing the “interpretation” for the  $480^\circ$  angle in the statement:

**Claim —** Point  $A_1$  is the circumcenter of  $\triangle A_2BC$ , and similarly for the others.

*Proof.* We have  $\angle BA_1C = 180^\circ - 2\alpha$ , and

$$\begin{aligned} \angle BA_2C &= 180^\circ - \angle CBC_1 - \angle B_1CB \\ &= 180^\circ - (60^\circ - \gamma) - (60^\circ - \beta) \\ &= 60^\circ + \beta + \gamma = 90^\circ - \alpha = \frac{1}{2} \angle BA_1C. \end{aligned}$$

Since  $A_1$  lies inside  $\triangle BA_2C$ , it follows  $A_1$  is exactly the circumcenter.  $\square$

**Claim —** Quadrilateral  $B_2C_1B_1C$  can be inscribed in a circle, say  $\gamma_a$ . Circles  $\gamma_b$  and  $\gamma_c$  can be defined similarly. Finally, these three circles are pairwise distinct.

*Proof.* Using directed angles now, we have

$$\angle B_2B_1C_2 = 180^\circ - \angle AB_1B_2 = 180^\circ - 2\angle ACB = 180^\circ - 2(60^\circ - \alpha) = 60^\circ + 2\alpha.$$

By the same token,  $\angle B_2C_1C_2 = 60^\circ + 2\alpha$ . This establishes the existence of  $\gamma_a$ .

The proof for  $\gamma_b$  and  $\gamma_c$  is the same. Finally, to show the three circles are distinct, it would be enough to verify that the convex hexagon  $A_2C_1B_2A_1C_2B_1$  is not cyclic.

Assume for contradiction it was cyclic. Then

$$360^\circ = \angle C_2 A_1 B_1 + \angle B_2 C_1 A_2 + \angle A_2 B_1 C_2 = \angle B A_1 C + \angle C B_1 A + \angle A C_1 B = 480^\circ$$

which is absurd. This contradiction eliminates the degenerate case, so the three circles are distinct.  $\square$

For the remainder of the solution, let  $\text{Pow}(P, \omega)$  denote the power of a point  $P$  with respect to a circle  $\omega$ .

Let line  $AA_1$  meet  $\gamma_b$  and  $\gamma_c$  again at  $X$  and  $Y$ , and set  $k_a := \frac{AX}{AY}$ . Consider the locus of all points  $P$  such that

$$\mathcal{C}_a := \left\{ \text{points } P \text{ in the plane satisfying } \text{Pow}(P, \gamma_b) = k_a \text{Pow}(P, \gamma_c) \right\}.$$

We recall the *coaxiality lemma*<sup>1</sup>, which states that (given  $\gamma_b$  and  $\gamma_c$  are not concentric) the locus  $\mathcal{C}_a$  must be either a circle (if  $k_a \neq 1$ ) or a line (if  $k_a = 1$ ).

On the other hand,  $A_1$ ,  $A_2$ , and  $A$  all obviously lie on  $\mathcal{C}_a$ . (For  $A_1$  and  $A_2$ , the powers are both zero, and for the point  $A$ , we have  $\text{Pow}(P, \gamma_b) = AX \cdot AA_1$  and  $\text{Pow}(P, \gamma_c) = AY \cdot AA_1$ .) So  $\mathcal{C}_a$  must be exactly the circumcircle of  $\triangle AA_1 A_2$  from the problem statement.

We turn to evaluating  $k_a$  more carefully. First, note that

$$\angle A_1 X B_1 = \angle A_1 B_2 B_1 = \angle C B_2 B_1 = 90^\circ - \angle B_2 A C = 90^\circ - (60^\circ - \gamma) = 30^\circ + \gamma.$$

Now using the law of sines, we derive

$$\begin{aligned} \frac{AX}{AB_1} &= \frac{\sin \angle A B_1 X}{\sin \angle A X B_1} = \frac{\sin(\angle A_1 X B_1 - \angle X A B_1)}{\sin \angle A_1 X B_1} \\ &= \frac{\sin((30^\circ + \gamma) - (30^\circ - \beta))}{\sin(30^\circ + \gamma)} = \frac{\sin(\beta + \gamma)}{\sin(30^\circ + \gamma)}. \end{aligned}$$

Similarly,  $AY = AC_1 \cdot \frac{\sin(\beta + \gamma)}{\sin(30^\circ + \beta)}$ , so

$$k_a = \frac{AX}{AY} = \frac{AB_1}{AC_1} \cdot \frac{\sin(30^\circ + \beta)}{\sin(30^\circ + \gamma)}.$$

Now define analogous constants  $k_b$  and  $k_c$  and circles  $\mathcal{C}_b$  and  $\mathcal{C}_c$ . Owing to the symmetry of the expressions, we have the key relation

$$k_a k_b k_c = 1.$$

In summary, the three circles in the problem statement may be described as

$$\begin{aligned} \mathcal{C}_a &= (AA_1 A_2) = \{ \text{points } P \text{ such that } \text{Pow}(P, \gamma_b) = k_a \text{Pow}(P, \gamma_c) \} \\ \mathcal{C}_b &= (BB_1 B_2) = \{ \text{points } P \text{ such that } \text{Pow}(P, \gamma_c) = k_b \text{Pow}(P, \gamma_a) \} \\ \mathcal{C}_c &= (CC_1 C_2) = \{ \text{points } P \text{ such that } \text{Pow}(P, \gamma_a) = k_c \text{Pow}(P, \gamma_b) \}. \end{aligned}$$

Since  $k_a, k_b, k_c$  have product 1, it follows that any point on at least two of the circles must lie on the third circle as well. The convexity of hexagon  $A_2 C_1 B_2 A_1 C_2 B_1$  mentioned earlier ensures these any two of these circles do intersect at two different points, completing the solution.

<sup>1</sup>We quickly outline a proof of this lemma: in the Cartesian coordinate system, the expression  $\text{Pow}((x, y), \omega)$  is an expression of the form  $x^2 + y^2 + \bullet x + \bullet y + \bullet$  for some constants  $\bullet$  whose value does not matter. Substituting this into the equation  $\frac{k_a \text{Pow}(P, \gamma_c) - \text{Pow}(P, \gamma_b)}{k_a - 1} = 0$  gives the equation of a circle provided  $k_a \neq 1$ , and when  $k_a = 1$ , one instead recovers the radical axis.