Team Selection Test for the 64$^{\text{th}}$ International Mathematical Olympiad

United States of America

Day I

Thursday, December 8, 2022

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, December 12 at noon Eastern time.

IMO TST 1. There are 2022 equally spaced points on a circular track γ of circumference 2022. The points are labeled A_1, A_2, $\ldots$, A_{2022} in some order, each label used once. Initially, Bunbun the Bunny begins at A_1. She hops along γ from A_1 to A_2, then from A_2 to A_3, until she reaches A_{2022}, after which she hops back to A_1. When hopping from P to Q, she always hops along the shorter of the two arcs $\widehat{PQ}$ of γ; if $\overline{PQ}$ is a diameter of γ, she moves along either semicircle.

Determine the maximal possible sum of the lengths of the 2022 arcs which Bunbun traveled, over all possible labellings of the 2022 points.

IMO TST 2. Let ABC be an acute triangle. Let M be the midpoint of side BC, and let E and F be the feet of the altitudes from B and C, respectively. Suppose that the common external tangents to the circumcircles of triangles BME and CMF intersect at a point K, and that K lies on the circumcircle of ABC. Prove that line AK is perpendicular to line BC.

IMO TST 3. Consider pairs (f,g) of functions from the set of nonnegative integers to itself such that

- $f(0) \geq f(1) \geq f(2) \geq \cdots \geq f(300) \geq 0$;
- $f(0) + f(1) + f(2) + \cdots + f(300) \leq 300$;
- for any 20 nonnegative integers n_1, n_2, $\ldots$, n_{20}, not necessarily distinct, we have

$$g(n_1 + n_2 + \cdots + n_{20}) \leq f(n_1) + f(n_2) + \cdots + f(n_{20}).$$

Determine the maximum possible value of $g(0) + g(1) + \cdots + g(6000)$ over all such pairs of functions.

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Day II

Thursday, January 12, 2023

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, January 16 at noon Eastern time.

IMO TST 4. Let $\lfloor \bullet \rfloor$ denote the floor function. For nonnegative integers a and b, their *bitwise xor*, denoted $a \oplus b$, is the unique nonnegative integer such that

$$\left\lfloor \frac{a}{2^k} \right\rfloor + \left\lfloor \frac{b}{2^k} \right\rfloor - \left\lfloor \frac{a \oplus b}{2^k} \right\rfloor$$

is even for every integer $k \geq 0$. (For example, $9 \oplus 10 = 1001_2 \oplus 1010_2 = 0011_2 = 3$.)

Find all positive integers a such that for any integers $x > y \geq 0$, we have

$$x \oplus ax \neq y \oplus ay.$$

IMO TST 5. Let m and n be fixed positive integers. Tsvety and Freyja play a game on an infinite grid of unit square cells. Tsvety has secretly written a real number inside of each cell so that the sum of the numbers within every rectangle of size either $m \times n$ or $n \times m$ is zero. Freyja wants to learn all of these numbers.

One by one, Freyja asks Tsvety about some cell in the grid, and Tsvety truthfully reveals what number is written in it. Freyja wins if, at any point, Freyja can simultaneously deduce the number written in every cell of the entire infinite grid. (If this never occurs, Freyja has lost the game and Tsvety wins.)

In terms of m and n, find the smallest number of questions that Freyja must ask to win, or show that no finite number of questions can suffice.

IMO TST 6. Let $\mathbb{N}$ denote the set of positive integers. Fix a function $f\colon \mathbb{N} \to \mathbb{N}$ and for any $m, n \in \mathbb{N}$ define

$$\Delta(m,n) = \underbrace{f(f(\ldots f}_{f(n)\text{ times}}(m)\ldots)) - \underbrace{f(f(\ldots f}_{f(m)\text{ times}}(n)\ldots)).$$

Suppose $\Delta(m,n) \neq 0$ for any distinct $m, n \in \mathbb{N}$. Show that Δ is unbounded, meaning that for any constant C there exist $m, n \in \mathbb{N}$ with $|\Delta(m,n)| > C$.