IMO 2021 Solution Notes

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This is a compilation of solutions for the 2021 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Let $n \geq 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n + 1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

2. Show that the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i - x_j|} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i + x_j|}$$

   holds for all real numbers x_1, x_2, ..., x_n.

3. Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

4. Let Γ be a circle with center I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

5. Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the kth move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

   Prove that there exists a value of k such that, on the kth move, Jumpy swaps some walnuts a and b such that $a < k < b$.

6. Let $m \geq 2$ be an integer, A a finite set of integers (not necessarily positive) and B_1, B_2, ..., B_m subsets of A. Suppose that, for every $k = 1, 2, \ldots, m$, the sum of the elements of B_k is m^k. Prove that A contains at least $\frac{m}{2}$ elements.

§1 Solutions to Day 1

§1.1 IMO 2021/1, proposed by Australia

Available online at https://aops.com/community/p22698392.

> **Problem statement**
>
> Let $n \geq 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

We will find three cards $a < b < c$ such that

$$b + c = (2k+1)^2$$
$$c + a = (2k)^2$$
$$a + b = (2k-1)^2$$

for some integer k. Solving for a, b, c gives

$$a = \frac{(2k)^2 + (2k-1)^2 - (2k+1)^2}{2} = 2k^2 - 4k$$
$$b = \frac{(2k+1)^2 + (2k-1)^2 - (2k)^2}{2} = 2k^2 + 1$$
$$c = \frac{(2k+1)^2 + (2k)^2 - (2k-1)^2}{2} = 2k^2 + 4k$$

We need to show that when $n \geq 100$, one can find a suitable k.

Let

$$I_k := \{n \in \mathbb{Z} \mid n \leq a < b < c \leq 2n\}$$
$$= \{n \in \mathbb{Z} \mid k^2 + 2k \leq n \leq 2k^2 - 4k\}$$

be the interval such that when $n \in I_k$, the problem dies for that choice of k. It would be sufficient to show these intervals I_k cover all the integers ≥ 100. Starting from $I_9 = \{99 \leq n \leq 126\}$, we have

$$k \geq 9 \implies 2k^2 - 4k \geq (k+1)^2 + 2(k+1)$$

which means the right endpoint of I_k exceeds the left endpoint of I_{k+1}. Hence for $n \geq 99$ in fact the problem is true.

> **Remark.** The problem turns out to be false for $n = 98$, surprisingly. The counterexample is for one pile to be
>
> $$\{98, 100, 102, \ldots, 126\} \cup \{129, 131, 135, \ldots, 161\} \cup \{162, 164, \ldots, 196\}.$$

§1.2 IMO 2021/2, proposed by Calvin Deng

Available online at https://aops.com/community/p22697952.

> **Problem statement**
>
> Show that the inequality
>
> $$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \le \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$
>
> holds for all real numbers x_1, x_2, ..., x_n.

The proof is by induction on $n \ge 1$ with the base cases $n = 1$ and $n = 2$ being easy to verify by hand.

In the general situation, consider replacing the tuple $(x_i)_i$ with $(x_i + t)_i$ for some parameter $t \in \mathbb{R}$. The inequality becomes

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \le \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j + 2t|}.$$

The left-hand side is independent of t.

> **Claim** — The right-hand side, viewed as a function $F(t)$ of t, is minimized when $2t = -(x_i + x_j)$ for some i and j.

Proof. Since $F(t)$ is the sum of piecewise concave functions, it is hence itself piecewise concave. Moreover F increases without bound if $|t| \to \infty$.

On each of the finitely many intervals on which $F(t)$ is concave, the function is minimized at its endpoints. Hence the minimum value must occur at one of the endpoints. $\qquad\square$

If $t = -x_i$ for some i, this is the same as shifting all the variables so that $x_i = 0$. In that case, we may apply induction on $n - 1$ variables, deleting the variable x_i.

If $t = -\frac{x_i + x_j}{2}$, then notice

$$x_i + t = -(x_j + t)$$

so it's the same as shifting all the variables such that $x_i = -x_j$. In that case, we may apply induction on $n - 2$ variables, after deleting x_i and x_j.

§1.3 IMO 2021/3, proposed by Mykhalio Shtandenko (UKR)

Available online at https://aops.com/community/p22698068.

> **Problem statement**
>
> Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

This problem and solution were contributed by Abdullahil Kafi.

> **Claim —** Quadrilateral $BCEF$ is cyclic.

Proof. Let D' be the isogonal conjugate of the point D. The angle condition implies quadrilateral $CEDD'$ and $BFDD'$ are cyclic. By power of point we have

$$AE \cdot AC = AD \cdot AD' = AF \cdot AB$$

So $BCEF$ is cyclic. $\qquad\square$

> **Claim —** Line ZD is tangent to the circles (BCD) and (DEF) where $Z = EF \cap BC$.

Proof. Let $\angle CAD = \angle BAD = \alpha$, $\angle BCD = \beta$, $\angle DBC = \gamma$, $\angle ACD = \phi$, $\angle ABD = \epsilon$. From $\triangle ABC$ we have $2\alpha + \beta + \gamma + \phi + \epsilon = 180°$. Let ℓ be a line tangent to (BCD) and K be a point on it in the same side of AD as C and $L = AD \cap BC$. From our labeling we have,

$$\angle AFE = \beta + \phi \qquad \angle BFD = \alpha + \gamma \qquad \angle DFE = \alpha + \phi \qquad \angle CDL = \alpha + \phi$$

Now $\angle CDJ = 180° - \gamma - \beta - (\alpha + \phi) = \alpha + \epsilon$. So $\angle DFE = \angle EDK = \alpha + \epsilon$, which means ℓ is also tangent to (DEF). Now by the radical center theorem we have ℓ passes through Z. $\qquad\square$

Let M be the Miquel point of the cyclic quadrilateral $BCEF$. From the Miquel configuration we have A, M, Z are collinear and $(AFEM)$, $(ZCEM)$ are cyclic.

> **Claim —** Points B, X, M, E are cyclic.

Proof. Notice that $\angle EMB = 180° - \angle AMB - \angle EMZ = 180° - 2\angle ACB = \angle EXB$. $\qquad\square$

Let N be the other intersection of circles (ACD) and (DEX) and let R be the intersection of AC and BM.
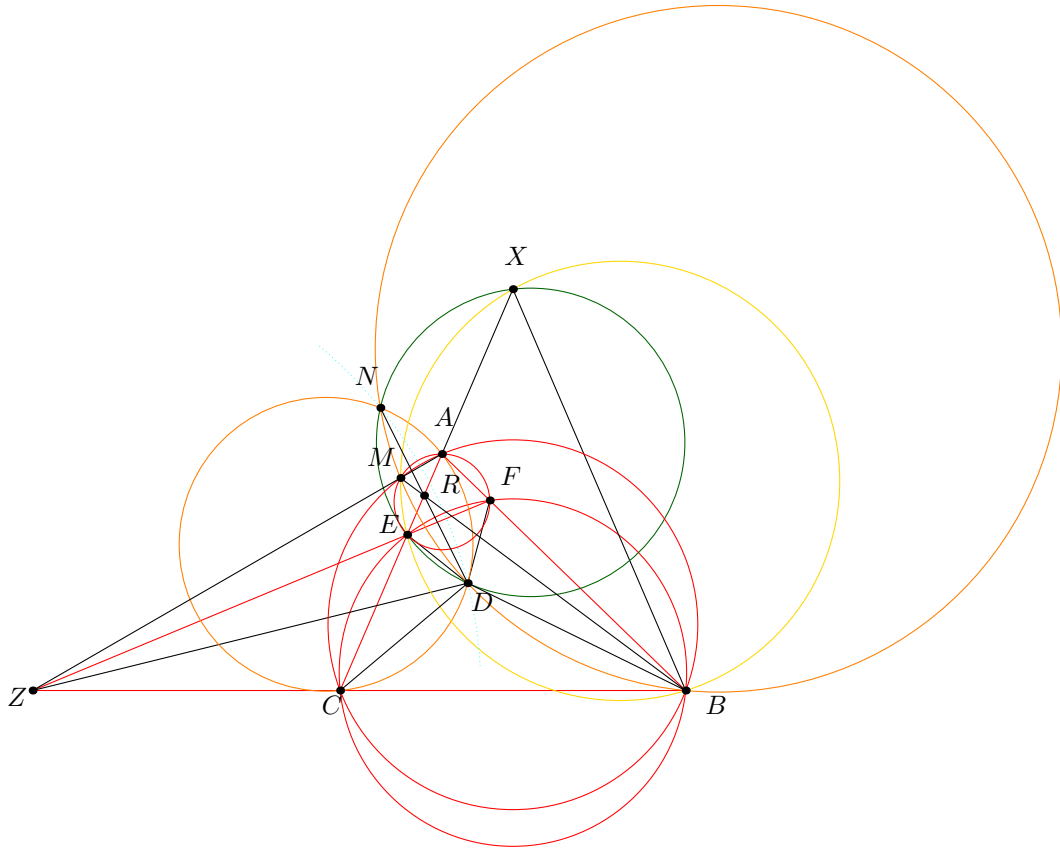
> **Claim —** Points B, D, M, N are cyclic.

Proof. By power of point we have

$$\operatorname{Pow}(R, (ACD)) = RC \cdot RA = RM \cdot RB = RE \cdot RX = \operatorname{Pow}(R, (DEX)).$$

Hence R lies on the radical axis of (ACD) and (DEX), so N, R, D are collinear. Also

$$RN \cdot RD = RA \cdot RC = RM \cdot RB$$

So $BDMN$ is cyclic. $\qquad\square$

Notice that (ACD), $(BDMN)$, (DEX) are coaxial so their centers are collinear. Now we just need to prove the centers of (ACD), $(BDMN)$ and Z are collinear. To prove this, take a circle ω with radius ZD centered at Z. Notice that by power of point

$$ZC \cdot ZB = ZD^2 = ZE \cdot ZF = ZM \cdot ZA$$

which means inversion circle ω swaps (ACD) and $(BDMN)$. So the centers of (ACD) and $(BDMN)$ must have to be collinear with the center of inversion circle, as desired.

§2 Solutions to Day 2

## §2.1 IMO 2021/4, proposed by Dominik Burek (POL) and Tomasz Ciesla (POL)

Available online at https://aops.com/community/p22698001.

> **Problem statement**
>
> Let Γ be a circle with center I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that
>
> $$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Let $PQRS$ be the contact points of Γ an $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$.
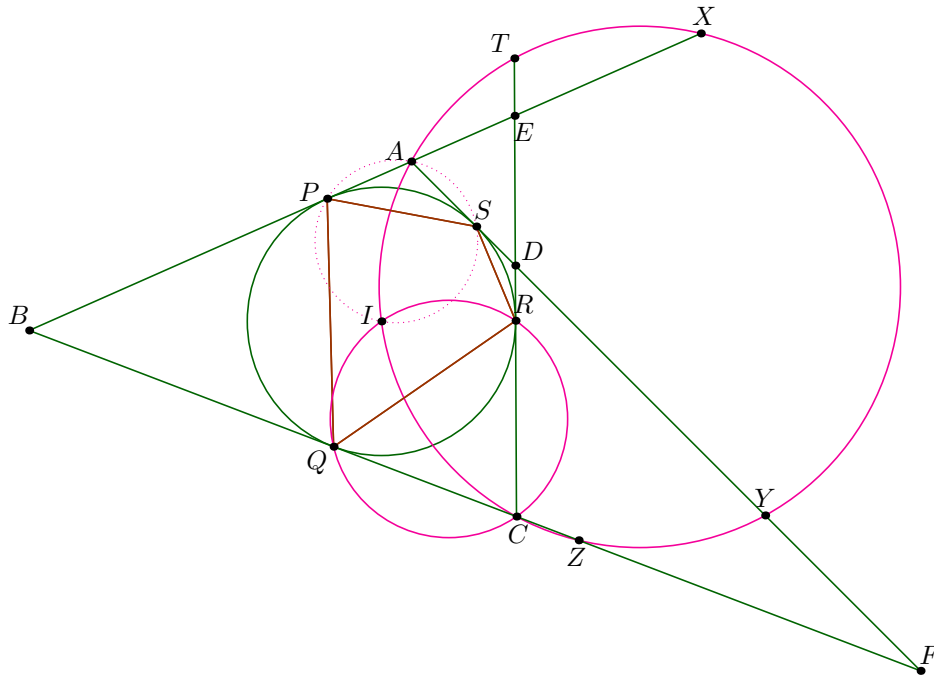
> **Claim —** We have $\triangle IQZ \cong \triangle IRT$. Similarly, $\triangle IPX \cong \triangle ISY$.

Proof. By considering $(CQIR)$ and $(CITZ)$, there is a spiral similarity similarity mapping $\triangle IQZ$ to $\triangle IRT$. Since $IQ = IR$, it is in fact a congruence. □

This congruence essentially solves the problem. First, it implies:

> **Claim —** $TX = YZ$.

Proof. Because we saw $IX = IY$ and $IT = IZ$. □

Then, we can compute

$$AD + DT + XA = AD + (RT - RD) + (XP - AP)$$
$$= (AD - RD - AP) + RT + XP = RT + XP$$

and

$$CD + DY + ZC = CD + (SY - SD) + (ZQ - QC)$$
$$= (CD - SD - QC) + SY + ZQ = SY + ZQ$$

but $ZQ = RT$ and $XP = SY$, as needed.

§2.2 IMO 2021/5, proposed by Spain

*Available online at https://aops.com/community/p22697921.*

Problem statement

Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the kth move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the kth move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Assume for contradiction no such k exists. We will use a so-called "threshold trick".

This process takes exactly 2021 steps. Right after the kth move, we consider a situation where we color walnut k red as well, so at the kth step there are k ones. For brevity, a non-red walnut is called black. An example is illustrated below with 2021 replaced by 6.
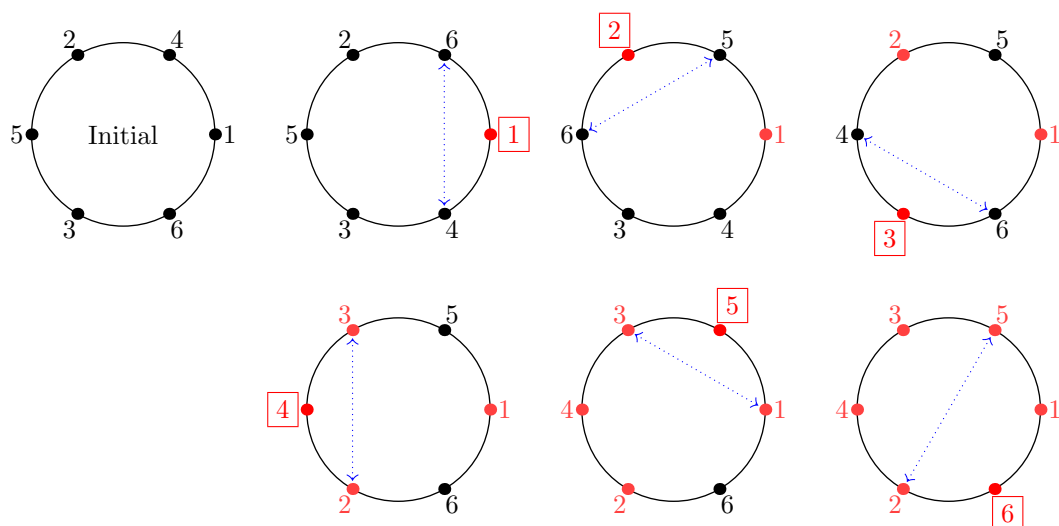
> **Claim —** At each step, the walnut that becomes red is between two non-red or two red walnuts.

Proof. By definition. □

On the other hand, if there are 2021 walnuts, one obtains a parity obstruction to this simplified process:

> **Claim —** After the first step, there is always a consecutive block of black walnuts positive even length.

Proof. After the first step, there is a block of 2020 black walnuts.

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Thereafter, note that a length 2 block of black walnuts can never be changed. Meanwhile for even lengths at least 4, if one places a red walnut inside it, the even length block splits into an odd length block and an even length block. □

> **Remark.** The statement is true with 2021 replaced by any odd number, and false for any even number.
>
> The motivation comes from the following rephrasing of the problem:
>
> > Start with all 0's and at each step change a 0 between two matching numbers from a 0 to a 1.
>
> Although the coloring (or 0/1) argument may appear to lose information at first, I think it should be *equivalent* to the original process; the "extra" information comes down to the choice of which walnut to color red at each step.

§2.3 IMO 2021/6, proposed by Austria

Available online at https://aops.com/community/p22698082.

> **Problem statement**
>
> Let $m \geq 2$ be an integer, A a finite set of integers (not necessarily positive) and B_1, B_2, ..., B_m subsets of A. Suppose that, for every $k = 1, 2, \ldots, m$, the sum of the elements of B_k is m^k. Prove that A contains at least $\frac{m}{2}$ elements.

If $0 \leq X < m^{m+1}$ is a multiple of m, then write it in base m as

$$X = \sum_{i=1}^{m} c_i m^i \qquad c_i \in \{0, 1, 2, \ldots, m-1\}$$

Then swapping the summation to over A through the B_i's gives

$$X = \sum_{i=1}^{n} \left(\sum_{b \in B_i} b \right) c_i = \sum_{a \in A} f_a(X) a \quad \text{where} \quad f_a(X) := \sum_{i:a \in B_i} c_i.$$

Evidently, $0 \leq f_a(X) \leq n(m-1)$ for any a and X. So, setting $|A| = n$, the right-hand side of the display takes on at most $(n(m-1) + 1)^n$ distinct values. This means

$$m^m \leq (n(m-1))^n$$

which implies $n \geq m/2$.

> **Remark** (Motivation comments from USJL)**.** In linear algebra terms, we have some n-dimensional 0/1 vectors $\vec{v_1}$, ..., $\vec{v_m}$ and an n-dimensional vector $\vec{a}$ such that $\vec{v_i} \cdot \vec{a} = m^i$ for $i = 1, \ldots, m$. The intuition is that if n is too small, then there should be lots of linear dependences between $\vec{v_i}$.
>
> In fact, *Siegel's lemma* is a result that says, if there are many more vectors than the dimension of the ambient space, there exist linear dependences whose coefficients are not-too-big integers. On the other hand, any linear dependence between m, m^2, ..., m^m is going to have coefficients that are pretty big; at least one of them needs to exceed m.
>
> Applying Siegel's lemma turns out to solve the problem (and is roughly equivalent to the solution above).

> **Remark.** In https://aops.com/community/p23185192, `dgrozev` shows the stronger bound $n \geq \left(\frac{2}{3} + \frac{c}{\log m} \right) m$ elements, for some absolute constant $c > 0$.