Team Selection Test for the 62^{nd} International Mathematical Olympiad

United States of America

February 25, 2021

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, March 1 at noon Eastern time.

IMO TST 1. Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

IMO TST 2. Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D. Let O and ρ denote the circumcenter and circumradius of $\triangle XCD$, respectively.

Prove there exists a fixed point K and a real number c, independent of X, for which $OK^2 - \rho^2 = c$ always holds regardless of the choice of X.

IMO TST 3. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z-y}{z-x}f(x) + \frac{y-x}{z-x}f(z)\right) \leq f\left(\frac{x+z}{2}\right) - \frac{f(x)+f(z)}{2}$$

for all real numbers $x < y < z$.