

IMO 2019 Solution Notes

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This is an compilation of solutions for the 2019 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Solve over \mathbb{Z} the functional equation $f(2a) + 2f(b) = f(f(a + b))$.
2. In triangle ABC point A_1 lies on side BC and point B_1 lies on side AC . Let P and Q be points on segments AA_1 and BB_1 , respectively, such that $\overline{PQ} \parallel \overline{AB}$. Point P_1 is chosen on ray PB_1 beyond B_1 such that $\angle PP_1C = \angle BAC$. Point Q_1 is chosen on ray QA_1 beyond A_1 such that $\angle CQ_1Q = \angle CBA$. Prove that points P_1, Q_1, P, Q are cyclic.
3. A social network has 2019 users, some pairs of which are friends (friendship is symmetric). If A, B, C are three users such that AB are friends and AC are friends but BC is not, then the administrator may perform the following operation: change the friendships such that BC are friends, but AB and AC are no longer friends.
Initially, 1009 users have 1010 friends and 1010 users have 1009 friends. Prove that the administrator can make a sequence of operations such that all users have at most 1 friend.
4. Solve over positive integers the equation

$$k! = \prod_{i=0}^{n-1} (2^n - 2^i) = (2^n - 1)(2^n - 2)(2^n - 4) \dots (2^n - 2^{n-1}).$$

5. Let n be a positive integer. Harry has n coins lined up on his desk, which can show either heads or tails. He does the following operation: if there are k coins which show heads and $k > 0$, then he flips the k th coin over; otherwise he stops the process. (For example, the process starting with THT would be $THT \rightarrow HHT \rightarrow HTT \rightarrow TTT$, which takes three steps.)
Prove the process will always terminate, and determine the average number of steps this takes over all 2^n configurations.
6. Let ABC be a triangle with incenter I and incircle ω . Let D, E, F denote the tangency points of ω with $\overline{BC}, \overline{CA}, \overline{AB}$. The line through D perpendicular to \overline{EF} meets ω again at R (other than D), and line AR meets ω again at P (other than R). Suppose the circumcircles of $\triangle PCE$ and $\triangle PBF$ meet again at Q (other than P). Prove that lines DI and PQ meet on the external $\angle A$ -bisector.

§1 IMO 2019/1, proposed by Liam Baker (SAF)

Solve over \mathbb{Z} the functional equation $f(2a) + 2f(b) = f(f(a + b))$.

Notice that $f(x) \equiv 0$ or $f(x) \equiv 2x + k$ work and are clearly the only linear solutions. We now prove all solutions are linear.

Let $P(a, b)$ be the assertion.

Claim — For each $x \in \mathbb{Z}$ we have $f(2x) = 2f(x) - f(0)$.

Proof. Compare $P(0, x)$ and $P(x, 0)$. □

Now, $P(a, b)$ and $P(0, a + b)$ give

$$\begin{aligned} f(f(a + b)) &= f(2a) + 2f(b) = f(0) + 2f(a + b) \\ \implies [2f(a) - f(0)] + 2f(b) &= f(0) + 2f(a + b) \\ \implies (f(a) - f(0)) + (f(b) - f(0)) &= (f(a + b) - f(0)). \end{aligned}$$

Thus the map $x \mapsto f(x) - f(0)$ is additive, therefore linear.

Remark. The same proof works on the functional equation

$$f(2a) + 2f(b) = g(a + b)$$

where g is an arbitrary function (it implies that f is linear).

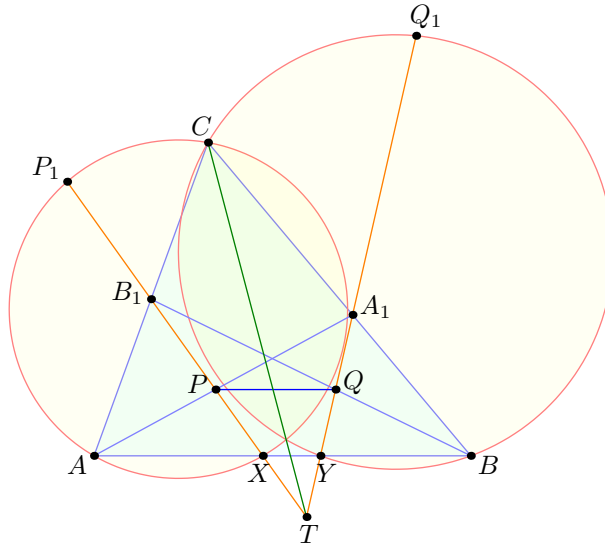
§2 IMO 2019/2, proposed by Anton Trygub (UKR)

In triangle ABC point A_1 lies on side BC and point B_1 lies on side AC . Let P and Q be points on segments AA_1 and BB_1 , respectively, such that $PQ \parallel AB$. Point P_1 is chosen on ray PB_1 beyond B_1 such that $\angle PP_1C = \angle BAC$. Point Q_1 is chosen on ray QA_1 beyond A_1 such that $\angle CQ_1Q = \angle CBA$. Prove that points P_1, Q_1, P, Q are cyclic.

We present two solutions.

First solution by bary (Evan Chen) Let PB_1 and QA_1 meet line AB at X and Y . Since $\overline{XY} \parallel \overline{PQ}$ it is equivalent to show P_1XYQ_1 is cyclic (Reim's theorem) Note that P_1CXA and Q_1CYB are cyclic.

Letting $T = \overline{PX} \cap \overline{QY}$ (possibly at infinity), it suffices to show that the radical axis of $\triangle CXA$ and $\triangle CYB$ passes through T , because that would imply P_1XYQ_1 is cyclic (by power of a point when T is Euclidean, and because it is an isosceles trapezoid if T is at infinity).



To this end we use barycentric coordinates on $\triangle ABC$. We begin by writing

$$P = (u + t : s : r), \quad Q = (t : u + s : r)$$

from which it follows that $A_1 = (0 : s : r)$ and $B_1 = (t : 0 : r)$.

Next, compute $X = (\det \begin{bmatrix} u+t & r \\ t & r \end{bmatrix} : \det \begin{bmatrix} s & r \\ 0 & r \end{bmatrix} : 0) = (u : s : 0)$. Similarly, $Y = (t : u : 0)$. So we have computed all points.

Claim — Line B_1X has equation $-rs \cdot x + ru \cdot y + st \cdot z = 0$, while line C_1Y has equation $ru \cdot x - rt \cdot y + st \cdot z = 0$.

Proof. Line B_1X is $0 = \det(B_1, X, -) = \det \begin{bmatrix} t & 0 & r \\ u & s & 0 \\ x & y & z \end{bmatrix}$. Line C_1Y is analogous. \square

Claim — The radical axis $(u + t)y - (u + s)x = 0$.

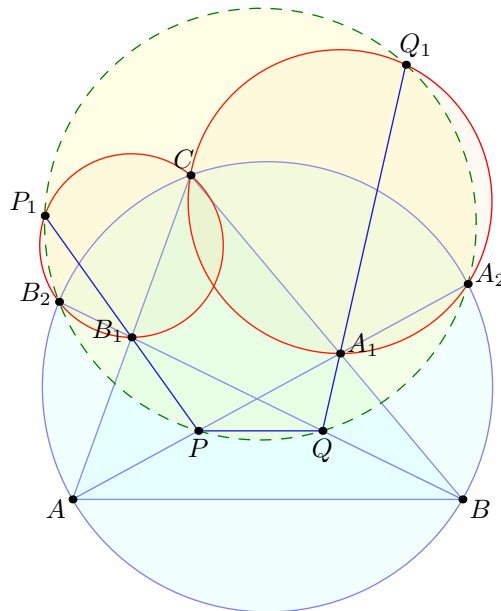
Proof. Circle (AXC) is given by $-a^2yz - b^2zx - c^2xy + (x + y + z) \cdot \frac{c^2 \cdot u}{u+s} y = 0$. Similarly, circle (BYC) has equation $-a^2yz - b^2zx - c^2xy + (x + y + z) \cdot \frac{c^2 \cdot u}{u+t} x = 0$. Subtracting gives the radical axis. \square

Finally, to see these three lines are concurrent, we now compute

$$\begin{aligned} \det \begin{bmatrix} -rs & ru & st \\ ru & -rt & st \\ -(u+s) & u+t & 0 \end{bmatrix} &= rst [[u(u+t) - t(u+s)] + [s(u+t) - u(u+s)]] \\ &= rst [(u^2 - st) + (st - u^2)] = 0. \end{aligned}$$

This completes the proof.

Second official solution by tricky angle chasing Let lines AA_1 and BB_1 meet at the circumcircle of $\triangle ABC$ again at points A_2 and B_2 . By Reim's theorem, PQA_2B_2 are cyclic.



Claim — The points P, Q, A_2, Q_1 are cyclic. Similarly the points P, Q, B_2, P_1 are cyclic.

Proof. Note that $CA_1A_2Q_1$ is cyclic since $\angle CQ_1A_1 = \angle CQ_1Q = \angle CBA = \angle CA_2A = \angle CA_2A_1$. Then $\angle QQ_1A_2 = \angle A_1Q_1A_2 = \angle A_1CA_2 = \angle BCA_2 = \angle BAA_2 = \angle QPA_2$. \square

This claim obviously solves the problem.

§3 IMO 2019/3, proposed by Adrian Beker (HRV)

A social network has 2019 users, some pairs of which are friends (friendship is symmetric). If A, B, C are three users such that AB are friends and AC are friends but BC is not, then the administrator may perform the following operation: change the friendships such that BC are friends, but AB and AC are no longer friends.

Initially, 1009 users have 1010 friends and 1010 users have 1009 friends. Prove that the administrator can make a sequence of operations such that all users have at most 1 friend.

We take the obvious graph formulation and call the move a *toggle*.

Claim — Let G be a connected graph. Then one can toggle G without disconnecting the graph, unless G is a clique, a cycle, or a tree.

Proof. Assume G is connected and not a tree, so it has a cycle. Take the smallest cycle C ; by hypothesis $C \neq G$.

If C is not a triangle (equivalently, G is triangle-free), then let $b \notin C$ be a vertex adjacent to C , say at a . Take a vertex c of the cycle adjacent to a (hence not to b). Then we can toggle abc .

Now assume there exists a triangle; let K be the maximal clique. By hypothesis, $K \neq G$. We take an edge $e = ab$ dangling off the clique, with $a \in K$ and $b \notin K$. Note some vertex c of K is not adjacent to b ; now toggle abc . \square

Back to the original problem; let G_{imo} be the given graph. The point is that we can apply toggles (by the claim) repeatedly, without disconnecting the graph, until we get a tree. This is because

- G_{imo} is connected, since any two vertices which are not adjacent have a common neighbor by pigeonhole ($1009 + 1009 + 2 > 2019$).
- G_{imo} cannot become a cycle, because it initially has an odd-degree vertex, and toggles preserve parity of degree!
- G_{imo} is obviously not a clique initially (and hence not afterwards).

So, we can eventually get G_{imo} to be a tree.

Once G_{imo} is a tree the problem follows by repeatedly applying toggles arbitrarily until no more are possible; the graph (although now disconnected) remains acyclic (in particular having no triangles) and therefore can only terminate in the desired situation.

Remark. Assume G_{imo} is connected. Then we have shown more strongly that if G_{imo} is not a clique, and has any vertex of odd degree, Noting that toggles preserve parity of degree, these conditions are actually necessary too. So this characterizes all connected graphs (and thus all graphs) for which the goal is possible.

§4 IMO 2019/4, proposed by Gabriel Chicas Reyes (SLV)

Solve over positive integers the equation

$$k! = \prod_{i=0}^{n-1} (2^n - 2^i) = (2^n - 1)(2^n - 2)(2^n - 4) \dots (2^n - 2^{n-1}).$$

The answer is $(n, k) = (1, 1)$ and $(n, k) = (2, 3)$ which work.

Let $A = \prod_i (2^n - 2^i)$, and assume $A = k!$ for some $k \geq 3$. Recall by exponent lifting that

$$\nu_3(2^t - 1) = \begin{cases} 0 & t \text{ odd} \\ 1 + \nu_3(t) & t \text{ even.} \end{cases}$$

Consequently, we can compute

$$k > \nu_2(k!) = \nu_2(A) = 1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2}$$

$$\frac{k}{3} \leq \nu_3(k!) = \nu_3(A) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{6} \right\rfloor + \dots < \frac{3}{4}n.$$

where the very first inequality can be justified say by Legendre's formula $\nu_2(k!) = k - s_2(k)$.

In this way, we get

$$\frac{9}{4}n > k > \frac{n(n-1)}{2}$$

which means $n \leq \frac{11}{2}$; a manual check then shows the solutions we claimed earlier are the only ones.

Remark. An amusing corollary of the problem pointed out in the Shortlist is that the symmetric group S_k cannot be isomorphic to the group $\text{GL}_n(\mathbb{F}_2)$ unless $(n, k) = (1, 1)$ or $(n, k) = (2, 3)$, which indeed produce isomorphisms.

§5 IMO 2019/5, proposed by David Altizio (USA)

Let n be a positive integer. Harry has n coins lined up on his desk, which can show either heads or tails. He does the following operation: if there are k coins which show heads and $k > 0$, then he flips the k th coin over; otherwise he stops the process. (For example, the process starting with THT would be $THT \rightarrow HHT \rightarrow HTT \rightarrow TTT$, which takes three steps.)

Prove the process will always terminate, and determine the average number of steps this takes over all 2^n configurations.

The answer is

$$E_n = \frac{1}{2}(1 + \dots + n) = \frac{1}{4}n(n+1)$$

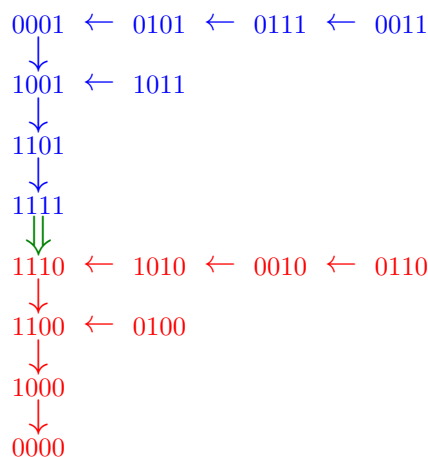
which is finite.

We'll represent the operation by a directed graph G_n on vertices $\{0, 1\}^n$ (each string points to its successor) with 1 corresponding to heads and 0 corresponding to tails. For $b \in \{0, 1\}$ we let $\bar{b} = 1 - b$, and denote binary strings as a sequence of n symbols.

The main claim is that G_n can be described explicitly in terms of G_{n-1} :

- We take two copies X and Y of G_{n-1} .
- In X , we take each string of length $n - 1$ and just append a 0 to it. In symbols, we replace $s_1 \dots s_{n-1} \mapsto s_1 \dots s_{n-1}0$.
- In Y , we toggle every bit, then reverse the order, and then append a 1 to it. In symbols, we replace $s_1 \dots s_{n-1} \mapsto \bar{s}_{n-1}\bar{s}_{n-2}\dots\bar{s}_11$.
- Finally, we add one new edge from Y to X by $11\dots1 \rightarrow 11\dots110$.

An illustration of G_4 is given below.



To prove this claim, we need only show the arrows of this directed graph remain valid. The graph X is correct as a subgraph of G_n , since the extra 0 makes no difference. As for Y , note that if $s = s_1 \dots s_{n-1}$ had k ones, then the modified string has $(n-1-k)+1 = n-k$ ones, ergo $\bar{s}_{n-1} \dots \bar{s}_1 1 \mapsto \bar{s}_{n-1} \dots \bar{s}_{k+1} s_k \bar{s}_{k-1} \dots \bar{s}_1 1$ which is what we wanted. Finally, the one edge from Y to X is obviously correct.

To finish, let E_n denote the desired expected value. Since $1 \dots 1$ takes n steps to finish we have

$$E_n = \frac{1}{2} [E_{n-1} + (E_{n-1} + n)]$$

based on cases on whether the chosen string is in X or Y or not. By induction, we have $E_n = \frac{1}{2}(1 + \dots + n) = \frac{1}{4}n(n+1)$, as desired.

Remark. Actually, the following is true: if the indices of the 1's are $1 \leq i_1 < \dots < i_\ell \leq 1$, then the number of operations required is

$$2(i_1 + \dots + i_\ell) - \ell^2.$$

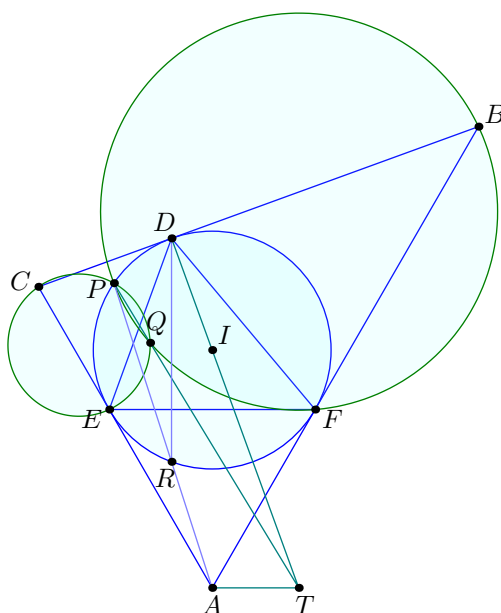
This problem also has an interpretation as a Turing machine: the head starts at a position on the tape (the binary string). If it sees a 1, it changes the cell to a 0 and moves left; if it sees a 0, it changes the cell to a 1 and moves right.

§6 IMO 2019/6, proposed by Anant Mudgal (IND)

Let ABC be a triangle with incenter I and incircle ω . Let D, E, F denote the tangency points of ω with $\overline{BC}, \overline{CA}, \overline{AB}$. The line through D perpendicular to \overline{EF} meets ω again at R (other than D), and line AR meets ω again at P (other than R). Suppose the circumcircles of $\triangle PCE$ and $\triangle PBF$ meet again at Q (other than P). Prove that lines DI and PQ meet on the external $\angle A$ -bisector.

We present three solutions.

First solution by complex numbers (Evan Chen, with Yang Liu) We use complex numbers with $D = x, E = y, F = z$.



Then $A = \frac{2yz}{y+z}, R = \frac{-yz}{x}$ and so

$$P = \frac{A - R}{1 - RA} = \frac{\frac{2yz}{y+z} + \frac{yz}{x}}{1 + \frac{yz}{x} \cdot \frac{2}{y+z}} = \frac{yz(2x + y + z)}{2yz + x(y + z)}.$$

We now compute

$$\begin{aligned}
O_B &= \det \begin{bmatrix} P & P\bar{P} & 1 \\ F & F\bar{F} & 1 \\ B & B\bar{B} & 1 \end{bmatrix} \div \det \begin{bmatrix} P & \bar{P} & 1 \\ F & \bar{F} & 1 \\ B & \bar{B} & 1 \end{bmatrix} = \det \begin{bmatrix} P & 1 & 1 \\ z & 1 & 1 \\ \frac{2xz}{x+z} & \frac{4xz}{(x+z)^2} & 1 \end{bmatrix} \div \det \begin{bmatrix} P & 1/P & 1 \\ z & 1/z & 1 \\ \frac{2xz}{x+z} & \frac{2}{x+z} & 1 \end{bmatrix} \\
&= \frac{1}{x+z} \det \begin{bmatrix} P & 0 & 1 \\ z & 0 & 1 \\ 2xz(x+z) & -(x-z)^2 & (x+z)^2 \end{bmatrix} \div \det \begin{bmatrix} P & 1/P & 1 \\ z & 1/z & 1 \\ 2xz & 2 & x+z \end{bmatrix} \\
&= \frac{(x-z)^2}{x+z} \cdot \frac{P-z}{(x+z)(P/z - z/P) + 2z - 2x + \frac{2xz}{P} - 2P} \\
&= \frac{(x-z)^2}{x+z} \cdot \frac{P-z}{\left(\frac{x}{z} - 1\right)P - 2(x-z) + (xz - z^2)\frac{1}{P}} \\
&= \frac{x-z}{x+z} \cdot \frac{P-z}{P/z + z/P - 2} = \frac{x-z}{x+z} \cdot \frac{P-z}{\frac{(P-z)^2}{Pz}} = \frac{x-z}{x+z} \cdot \frac{1}{\frac{1}{z} - \frac{1}{P}} \\
&= \frac{x-z}{x+z} \cdot \frac{yz(2x+y+z)}{y(2x+y+z) - (2yz+xy+xz)} = \frac{x-z}{x+z} \cdot \frac{yz(2x+y+z)}{xy + y^2 - yz - xz} \\
&= \frac{x-z}{x+z} \cdot \frac{yz(2x+y+z)}{(y-z)(x+y)}.
\end{aligned}$$

Similarly

$$O_C = \frac{x-y}{x+y} \cdot \frac{yz(2x+y+z)}{(z-y)(x+z)}.$$

Therefore, subtraction gives

$$O_B - O_C = \frac{yz(2x+y+z)}{(x+y)(x+z)(y-z)} [(x-z) + (x-y)] = \frac{yz(2x+y+z)(2x-y-z)}{(x+y)(x+z)(z-y)}.$$

It remains to compute T . Since $T \in \overline{ID}$ we have $t/x \in \mathbb{R}$ so $\bar{t} = t/x^2$. Also,

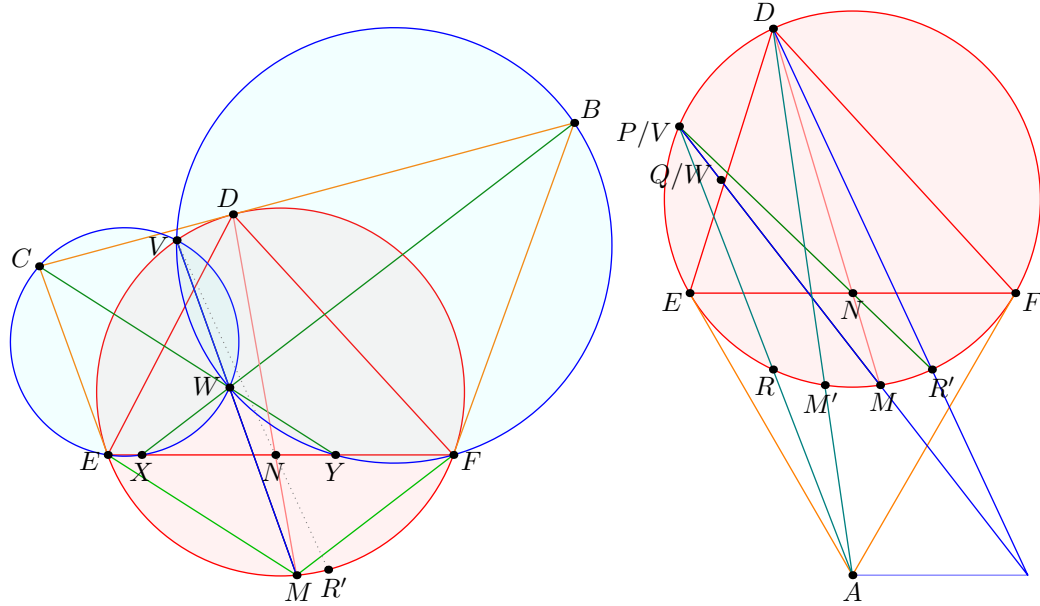
$$\begin{aligned}
\frac{t - \frac{2yz}{y+z}}{y+z} \in i\mathbb{R} &\implies 0 = \frac{t - \frac{2yz}{y+z}}{y+z} + \frac{\frac{t}{x^2} - \frac{2}{y+z}}{\frac{1}{y} + \frac{1}{z}} \\
&= \frac{1 + \frac{yz}{x^2}}{y+z} t - \frac{2yz}{(y+z)^2} - \frac{2yz}{(y+z)^2} \\
&\implies t = \frac{x^2}{x^2 + yz} \cdot \frac{4yz}{y+z}
\end{aligned}$$

Thus

$$\begin{aligned}
P - T &= \frac{yz(2x+y+z)}{2yz + x(y+z)} - \frac{4x^2yz}{(x^2 + yz)(y+z)} \\
&= yz \cdot \frac{(2x+y+z)(x^2 + yz)(y+z) - 4x^2(2yz + xy + xz)}{(y+z)(x^2 + yz)(2yz + xy + xz)} \\
&= -yz \cdot \frac{(2x-y-z)(x^2y + x^2z + 4xyz + y^2z + yz^2)}{(y+z)(x^2 + yz)(2yz + xy + xz)}.
\end{aligned}$$

This gives $\overline{PT} \perp \overline{O_BO_C}$ as needed.

Second solution by tethered moving points, with optimization (Evan Chen) Fix $\triangle DEF$ and ω , with $B = \overline{DD} \cap \overline{FF}$ and $C = \overline{DD} \cap \overline{EE}$. We consider a variable point M on ω and let X, Y be on \overline{EF} with $\overline{CY} \cap \parallel \overline{ME}$, $\overline{BX} \cap \parallel \overline{MF}$. We define $W = \overline{CY} \cap \overline{BX}$. Also, let line MW meet ω again at V .



Claim (Angle chasing) — Pentagons $CVWXE$ and $BVWYF$ are cyclic.

Proof. By $\angle EVW = \angle EVM = \angle EFM = \angle CEM = \angle ECW$ and $\angle EXW = \angle EFM = \angle CEM = \angle ECW$. □

Let $N = \overline{DM} \cap \overline{EF}$ and R' be the D -antipode on ω .

Claim (Black magic) — The points V, N, R' are collinear.

Proof. We use tethered moving points with $\triangle DEF$ fixed.

Obviously the map $\omega \mapsto \overline{EF} \mapsto \omega$ by $M \mapsto N \mapsto \overline{R'N} \cap \omega$ is projective. Also, the map $\omega \mapsto \overline{EF} \mapsto \omega$ by $M \mapsto X \mapsto V$ is also projective (the first by projection to the line at infinity at back; the second say by inversion at E).

So it suffices to check for three points. When $M = E$ we get $N = E$ so $\overline{R'N} \cap \omega = E$, while $W = E$ and thus $V = E$. The case $M = F$ is similar. Finally, if $M = R'$, then W is the center of ω and so $V = \overline{R'N} \cap \overline{EF} = D$. □

We now address the original problem by specializing M : choose it so that N is the midpoint of \overline{EF} . Let $M' = \overline{DA} \cap (DEF)$.

Claim — After this specialization, $V = P$ and $W = Q$.

Proof. Thus $\overline{RR'}$ and $\overline{MM'}$ are parallel to \overline{EF} . From $(EF; PR) = -1 = (EF; N\infty) \stackrel{R'}{=} (EF; NV)$, we derive that $P = V$ and $Q = R$, proving (i). □

Finally, the concurrence requested follows by Pascal theorem on $M'MDR'PR$.

Third solution by power of a point linearity (Luke Robitaille) Let us define

$$f(\bullet) = \text{Pow}(\bullet, (CPE)) - \text{Pow}(\bullet, (BPF))$$

which is a linear function from the plane to \mathbb{R} .

Define $W = \overline{BA} \cap \overline{PE}$, $V = \overline{AC} \cap \overline{PF}$. Also, let $W_1 = \overline{ER} \cap \overline{AB}$, $V_1 = \overline{FR} \cap \overline{AB}$. Note that

$$-1 = (PR; EF) \stackrel{E}{=} (WA; W_1F)$$

and similarly $(VA; V_1E) = -1$.

Claim — We have

$$\begin{aligned} f(F) &= \frac{|EF| \cdot (s - c) \sin C/2}{\sin B/2} \\ f(E) &= -\frac{|EF| \cdot (s - b) \sin B/2}{\sin C/2}. \end{aligned}$$

Proof. We have

$$f(W) = WF^2 - WB \cdot WF = WF \cdot BF$$

where lengths are directed. Next,

$$\begin{aligned} f(F) &= \frac{AF \cdot f(W) + FW \cdot f(A)}{AW} \\ &= \frac{AF \cdot WF \cdot BF + FW \cdot (AE \cdot AC - AF \cdot AB)}{AW} \\ &= \frac{WF(AF \cdot BF + AF \cdot AB) + FW \cdot AE \cdot AC}{AW} \\ &= \frac{WF \cdot AF^2 - WF \cdot AE \cdot AC}{AW} = \frac{WF}{AW} \cdot (AE^2 - AE \cdot AC) \\ &= \frac{WF}{AW} \cdot AE \cdot CE = -\frac{W_1F}{AW_1} \cdot AE \cdot CE. \end{aligned}$$

Since $\triangle DEF$ is acute, the point R lies inside $\triangle AEF$. Thus W_1 lies inside segment \overline{AF} and the ratio $\frac{W_1F}{AW_1}$ is positive. We now determine its value: by the ratio lemma

$$\begin{aligned} \frac{|W_1F|}{|AW_1|} &= \frac{|EF| \sin \angle W_1EF}{|AE| \sin \angle AEW_1} \\ &= \frac{|EF| \sin \angle REF}{|AE| \sin \angle AER} \\ &= \frac{|EF| \sin \angle RDF}{|AE| \sin \angle EDR} \\ &= \frac{|EF| \sin C/2}{|AE| \sin B/2}. \end{aligned}$$

Also, we have $AE \cdot CE < 0$ since E lies inside \overline{AC} . Hence

$$f(F) = -\frac{|EF| \sin C/2}{|AE| \sin B/2} \cdot AE \cdot CE = |EF| \cdot \frac{|CE| \sin B/2}{\sin C/2} = |EF| \cdot \frac{(s - c) \sin B/2}{\sin C/2}.$$

The calculation for $f(E)$ is similar, (noting the sign flips since f is anti-symmetric in terms of B and C). \square

Let $Z \in \overline{DI}$ with $\angle ZAI = 90^\circ$ be the point requested in the problem now. Our goal is to show $f(Z) = 0$. We assume WLOG that $AB < AC$, so $\frac{ZA}{EF} > 0$. Then

$$\begin{aligned} |ZA| &= |AI| \cdot \tan \angle AIZ \\ &= |AI| \cdot \tan \angle (\overline{AI}, \overline{DI}) \\ &= \frac{s-a}{\cos A/2} \cdot \tan(\overline{BC}, \overline{EF}) \\ &= \frac{s-a}{\cos A/2} \tan(B/2 - C/2). \end{aligned}$$

To this end we compute

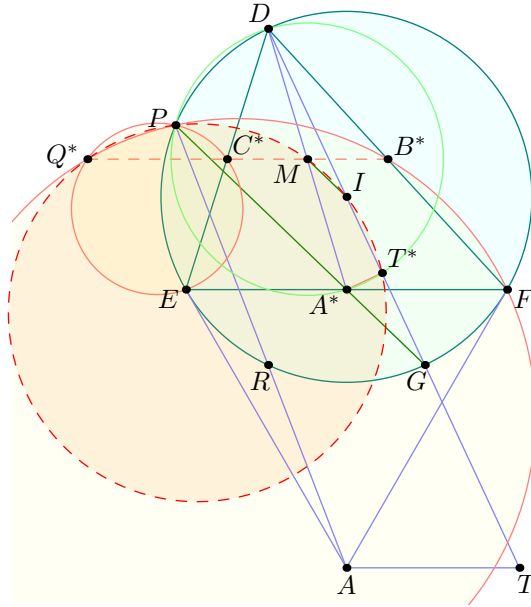
$$\begin{aligned} f(Z) &= f(A) + [f(Z) - f(A)] = f(A) + \frac{ZA}{EF} [f(E) - f(F)] \\ &= f(A) - \frac{ZA}{EF} \left[\frac{|EF| \cdot (s-b) \sin B/2}{\sin C/2} + \frac{|EF| \cdot (s-c) \sin C/2}{\sin B/2} \right] \\ &= f(A) - |ZA| \left[\frac{(s-b) \sin B/2}{\sin C/2} + \frac{(s-c) \sin C/2}{\sin B/2} \right] \\ &= [b(s-a) - c(s-a)] - |ZA| \left[\frac{(s-b) \sin B/2}{\sin C/2} + \frac{(s-c) \sin C/2}{\sin B/2} \right] \\ &= (b-c)(s-a) - \frac{s-a}{\cos A/2} \tan(B/2 - C/2) \left[\frac{(s-b) \sin B/2}{\sin C/2} + \frac{(s-c) \sin C/2}{\sin B/2} \right]. \end{aligned}$$

Dividing out,

$$\begin{aligned} \frac{f(Z)}{s-a} &= (b-c) - \frac{1}{\cos A/2} \tan(B/2 - C/2) \left[\frac{r \cos B/2}{\sin C/2} + \frac{r \cos C/2}{\sin B/2} \right] \\ &= (b-c) - \frac{r \tan(B/2 - C/2)}{\cos A/2} \cdot \frac{\cos B/2 \sin B/2 + \cos C/2 \sin C/2}{\sin C/2 \sin B/2} \\ &= (b-c) - \frac{r \tan(B/2 - C/2)}{\cos A/2} \cdot \frac{\sin B + \sin C}{2 \sin C/2 \sin B/2} \\ &= (b-c) - \frac{r \tan(B/2 - C/2)}{\cos A/2} \cdot \frac{\sin(B/2 + C/2) \cos(B/2 - C/2)}{\sin C/2 \sin B/2} \\ &= (b-c) - r \frac{\sin(B/2 - C/2)}{\sin B/2 \sin C/2} \\ &= (b-c) - r(\cot C/2 - \cot B/2) = (b-c) - ((s-c) - (s-b)) = 0. \end{aligned}$$

Fourth solution by incircle inversion (USA IMO live stream, led by Andrew Gu) Let T be the intersection of line DI and the external $\angle A$ -bisector. Also, let G be the antipode of D on ω .

We perform inversion around ω , using \bullet^* for the inverse. Then $\triangle A^*B^*C^*$ is the medial triangle of $\triangle DEF$, and T^* is the foot from A^* on to \overline{DI} . If we denote Q^* as the second intersection of (PC^*E) and (PB^*F) , then the goal is to show that Q^* lies on (PIT^*) .



Claim — Points Q^*, B^*, C^* are collinear.

Proof. $\angle PQ^*C^* = \angle PEC^* = \angle PED = \angle PFD = \angle PFB^* = \angle PQ^*B^*$. □

Claim (cf Brazil 2011/5) — Points P, A^*, G are collinear.

Proof. Project harmonic quadrilateral $PERF$ through G , noting $\overline{GR} \parallel \overline{EF}$. □

Denote by M the center of parallelogram $DC^*A^*B^*$. Note that it is the center of the circle with diameter $\overline{DA^*}$, which passes through P and T^* . Also, $\overline{MI} \parallel \overline{PA^*G}$.

Claim — Points P, M, I, T^* are cyclic.

Proof. $\angle IT^*P = \angle DT^*P = \angle DA^*P = \angle MA^*P = \angle A^*PM = \angle PMI$. □

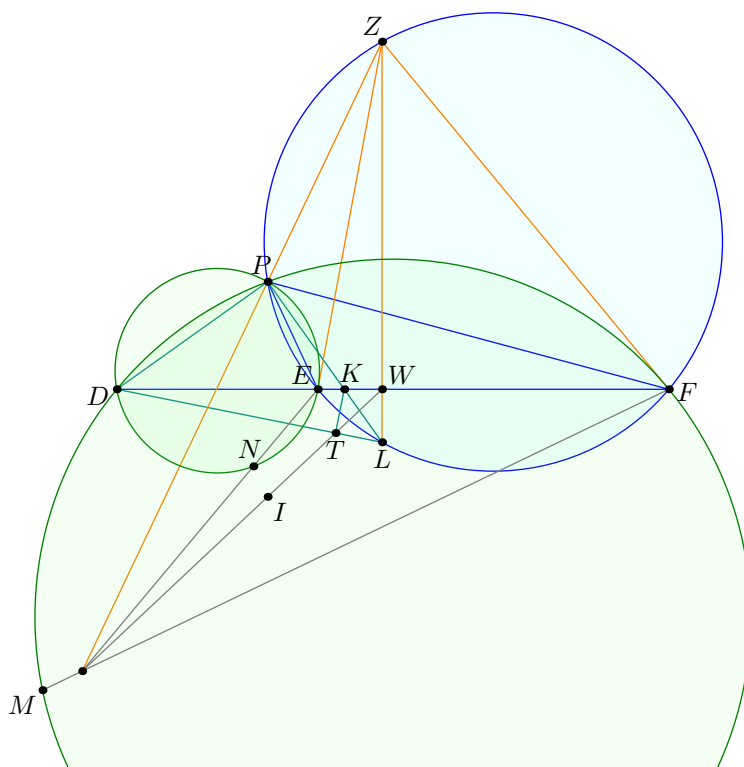
Claim — Points P, M, I, Q^* are cyclic.

Proof. $\angle MQ^*P = \angle C^*Q^*P = \angle C^*EP = \angle DEP = \angle DGP = \angle GPI = \angle MIP$. □

Fifth solution by double inversion (Brandon Wang, Luke Robitaille, Michael Ren, Evan Chen) We outline one final approach. After inverting about ω as in the previous approach, we then apply another inversion around P . Dropping the apostrophes/stars/etc now one can check that the problem we arrive at becomes the following.

Proposition (Doubly inverted problem)
 In $\triangle PEF$, the P -symmedian meets \overline{EF} and (PEF) at K, L . Let $D \in \overline{EF}$ with $\angle DPK = 90^\circ$, and let T be the foot from K to \overline{DL} . Denote by I the reflection of P about \overline{EF} . Finally, let $PDNE$ and $PDMF$ be cyclic harmonic quadrilaterals. Then lines EN, MF, TI , are concurrent.

The proof proceeds in three steps. Suppose the line through L perpendicular to \overline{EF} meets \overline{EF} at W and (PEF) at Z .



1. Since $\angle ZEP = \angle WLP = \angle WDP$, it follows \overline{ZE} is tangent to $(PDNE)$.
Similarly, \overline{ZF} is tangent to $(PDMF)$.
2. $\triangle WTP$ is the orthic triangle of $\triangle DKL$, so \overline{WD} bisects $\angle PWT$ and \overline{WTI} collinear.
3. $-1 = E(PN; DZ) = F(PM; DZ) = W(PI; DZ)$, so \overline{EN} , \overline{FM} , \overline{WI} meet on \overline{PZ} .