Team Selection Test for the 60^{th} International Mathematical Olympiad

United States of America

Day I

Thursday, December 6, 2018

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, December 10 at noon Eastern time.

IMO TST 1. Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

IMO TST 2. Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x)+x, \quad g(x)+2x, \quad \dots, \quad g(x)+100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

IMO TST 3. A *snake of length k* is an animal which occupies an ordered k-tuple $(s_1, \dots, s_k)$ of cells in an $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \dots, k-1$. If the snake is currently occupying $(s_1, \dots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can *move* to occupy $(s, s_1, \dots, s_{k-1})$ instead. The snake has *turned around* if it occupied $(s_1, s_2, \dots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \dots, s_1)$ instead.

Determine whether there exists an integer $n > 1$ such that one can place some snake of length at least $0.9n^2$ in an $n \times n$ grid which can turn around.

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Day II

Thursday, January 17, 2019

Time limit: 4.5 hours. Each problem is worth 7 points. You may keep the exam problems, but do not discuss them with anyone until Monday, January 21 at noon Eastern time.

IMO TST 4. We say a function $f\colon \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is *great* if for any nonnegative integers m and n,

$$f(m+1,n+1)f(m,n) - f(m+1,n)f(m,n+1) = 1.$$

If $A = (a_0, a_1, \dots)$ and $B = (b_0, b_1, \dots)$ are two sequences of integers, we write $A \sim B$ if there exists a great function f satisfying $f(n,0) = a_n$ and $f(0,n) = b_n$ for every nonnegative integer n (in particular, $a_0 = b_0$).

Prove that if A, B, C, and D are four sequences of integers satisfying $A \sim B$, $B \sim C$, and $C \sim D$, then $D \sim A$.

IMO TST 5. Let n be a positive integer. Tasty and Stacy are given a circular necklace with $3n$ sapphire beads and $3n$ turquoise beads, such that no three consecutive beads have the same color. They play a cooperative game where they alternate turns removing three consecutive beads, subject to the following conditions:

- Tasty must remove three consecutive beads which are turquoise, sapphire, and turquoise, in that order, on each of his turns.
- Stacy must remove three consecutive beads which are sapphire, turquoise, and sapphire, in that order, on each of her turns.

They win if all the beads are removed in $2n$ turns. Prove that if they can win with Tasty going first, they can also win with Stacy going first.

IMO TST 6. Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at point A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.