IMO 2018 Solution Notes

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September 30, 2020

This is an compilation of solutions for the 2018 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Let $\Gamma$ be the circumcircle of acute triangle $ABC$. Points $D$ and $E$ lie on segments $AB$ and $AC$, respectively, such that $AD = AE$. The perpendicular bisectors of $BD$ and $CE$ intersect the minor arcs $AB$ and $AC$ of $\Gamma$ at points $F$ and $G$, respectively. Prove that the lines $DE$ and $FG$ are parallel.

2. Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_n$ satisfying

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \ldots, n$, where indices are taken modulo $n$.

3. An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

\[
\begin{array}{ccc}
4 \\
2 & 6 \\
5 & 7 & 1 \\
8 & 3 & 10 & 9
\end{array}
\]

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + \cdots + 2018$?

4. A site is any point $(x, y)$ in the plane for which $x, y \in \{1, \ldots, 20\}$. Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones on unoccupied sites, with Amy going first; Amy has the additional restriction that no two of her stones may be at a distance equal to $\sqrt{5}$. They stop once either player cannot move. Find the greatest $K$ such that Amy can ensure that she places at least $K$ stones.

5. Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Suppose that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer for all but finitely many positive integers $n$. Prove that $(a_n)$ is eventually constant.

6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point $X$ lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$ 

Prove that $\angle BXA + \angle DXC = 180^\circ$. 

§1 IMO 2018/1, proposed by Silouanos Brazitikos, Vangelis Psyxas and Michael Sarantis (HEL)

Let $\Gamma$ be the circumcircle of acute triangle $ABC$. Points $D$ and $E$ lie on segments $AB$ and $AC$, respectively, such that $AD = AE$. The perpendicular bisectors of $BD$ and $CE$ intersect the minor arcs $AB$ and $AC$ of $\Gamma$ at points $F$ and $G$, respectively. Prove that the lines $DE$ and $FG$ are parallel.

We present a synthetic solution from the IMO shortlist as well as a complex numbers approach. We also outline a trig solution (the one I found at IMO)

**Synthetic solution (from Shortlist)** Construct parallelograms $AXFD$ and $AEGY$, noting that $X$ and $Y$ lie on $\Gamma$. As $XF \parallel AB$ we can let $M$ denote the midpoint of minor arcs $XF$ and $AB$ (which coincide). Define $N$ similarly.

Observe that $XF = AD = AE = YG$, so arcs $XF$ and $YG$ have equal measure; hence arcs $MF$ and $NG$ have equal measure; therefore $MN \parallel FG$.

Since $MN$ and $DE$ are both perpendicular to the $\angle A$ bisector, so we’re done.

**Complex numbers solution** Let $b, c, f, g, a$ be as usual. Note that

$$d - a = \left(2 \cdot \frac{f + a + b - abf}{2} - b\right) - a = f - \frac{ab}{f}$$

$$e - a = g - \frac{ac}{g}$$

We are given $AD = AE$ from which one deduces

$$\left(\frac{e - a}{d - a}\right)^2 = \frac{c}{b} \implies \left(\frac{g^2 - ac}{f^2 - ab}\right) = \frac{g^2c}{f^2b}$$

$$\implies bc(bg^2 - cf^2)a^2 = g^2f^4c - f^2g^4b = f^2g^2(f^2c - g^2b)$$

$$\implies bc \cdot a^2 = (fg)^2 \implies \left(-\frac{fg}{a}\right)^2 = bc.$$
Since $\frac{FG}{a}$ is the point $X$ on the circle with $AX \perp FG$, we conclude $FG$ is either parallel or perpendicular to the $\angle A$-bisector; it must the latter since the $\angle A$-bisector separates the two minor arcs.

**Trig solution (outline)** Let $\ell$ denote the $\angle A$ bisector. Fix $D$ and $F$. We define the phantom point $G'$ such that $FG' \perp \ell$ and $E'$ on side $AC$ such that $GE' = GC$.

**Claim (Converse of the IMO problem)** We have $AD = AE'$, so that $E = E'$.

**Proof.** Since $FG' \perp \ell$, one can deduce $\angle FBD = \frac{1}{2}C + x$ and $\angle GCA = \frac{1}{2}B + x$ for some $x$. (One fast way to see this is to note that $FG' \parallel MN$ where $M$ and $N$ are in the first solution.) Then $\angle FAB = \frac{1}{2}C - x$ and $\angle GAC = \frac{1}{2}B - x$.

Let $R$ be the circumradius. Now, by the law of sines,

$$BF = 2R \sin \left( \frac{1}{2}C - x \right).$$

From there we get

$$BD = 2 \cdot BF \cos \left( \frac{1}{2}C + x \right) = 4R \cos \left( \frac{1}{2}C + x \right) \sin \left( \frac{1}{2}C - x \right),$$

$$DA = AB - BD = 2R \sin C - 4R \cos \left( \frac{1}{2}C + x \right) \sin \left( \frac{1}{2}C - x \right)$$

$$= 2R \left[ \sin C - 2 \cos \left( \frac{1}{2}C + x \right) \sin \left( \frac{1}{2}C - x \right) \right]$$

$$= 2R \left[ \sin C - (\sin C - \sin 2x) \right] = 2R \sin 2x.$$

A similar calculation gives $AE' = 2R \sin 2x$ as needed. \qed

Thus, $FG' \parallel DE'$, so $G = G'$ as well. This concludes the proof.
§2 IMO 2018/2, proposed by Patrik Bak (SVK)

Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_n$ satisfying
\[ a_ia_{i+1} + 1 = a_{i+2} \]
for $i = 1, 2, \ldots, n$, where indices are taken modulo $n$.

The answer is $3 \mid n$, achieved by $(-1, -1, 2, -1, -1, 2, \ldots)$. We present two solutions.

First solution by inequalities

We compute $a_ia_{i+1}a_{i+2}$ in two ways:
\[ a_ia_{i+1}a_{i+2} = [a_{i+2} - 1]a_{i+2} = a_{i+2}^2 - a_{i+2} \]
\[ = a_i[a_{i+3} - 1] = a_ia_{i+3} - a_i. \]

Cyclically summing $a_{i+2}^2 - a_{i+2} = a_ia_{i+3} - a_i$ then gives
\[ \sum_i a_{i+2}^2 = \sum_i a_ia_{i+3} \iff \sum_{cyc} (a_i - a_{i+3})^2 = 0. \]

This means for inequality reasons the sequence is 3-periodic. Since the sequence is clearly not 1-periodic, as $x^2 + 1 = x$ has no real solutions. Thus $3 \mid n$.

Second solution by sign counting

Extend $a_n$ to be a periodic sequence. The idea is to look at the signs, and show the sequence of the signs must be $- - +$ repeated. This takes several steps:

- The pattern $- - -$ is impossible. Obvious, since the third term should be $> 1$.
- The pattern $++$ is impossible. Then the sequence becomes strictly increasing, hence may not be periodic.
- Zeros are impossible. If $a_1 = 0$, then $a_2 = 0, a_3 > 0, a_4 > 0$, which gives the impossible $++$.
- The pattern $- + - +$ is impossible. Compute the terms:
  \[ a_1 = -x < 0 \]
  \[ a_2 = -y < 0 \]
  \[ a_3 = 1 + xy > 1 \]
  \[ a_4 = 1 - y(1 + xy) < 0 \]
  \[ a_5 = 1 + (1 + xy)(1 - y(1 + xy)) < 1. \]

But now
\[ a_6 - a_5 = (1 + a_5a_4) - (1 + a_3a_4) = a_4(a_5 - a_3) > 0 \]
since $a_5 > 1 > a_3$. This means we have the impossible $++$ pattern.

- The infinite alternating pattern $- + - - + - + \ldots$ is impossible. Note that
  \[ a_1a_2 + 1 = a_3 < 0 < a_4 = 1 + a_2a_3 \implies a_1 < a_3 \]
since $a_2 > 0$; extending this we get $a_1 < a_3 < a_5 < \ldots$ which contradicts the periodicity.

We finally collate the logic of sign patterns. Since the pattern is not alternating, there must be $- -$ somewhere. Afterwards must be $+$, and then after that must be two minus signs (since one minus sign is impossible by impossibility of $- - + - -$ and $- - -$ is also forbidden); thus we get the periodic $- - +$ as desired.
§3 IMO 2018/3, proposed by Morteza Saghafian (IRN)

An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

```
  4
 2  6
 5  7  1
 8  3 10  9
```

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to \(1 + 2 + \cdots + 2018\)?

Let \(n = 2018\) and \(N = 1 + 2 + \cdots + n\). For every number \(d\) not in the bottom row, draw an arrow from \(d\) to the larger of the two numbers below it (i.e. if \(d = a - b\), draw \(d \rightarrow a\)). This creates an oriented forest (which looks like lightning strikes).

Consider the directed path starting from the top vertex \(A\). Starting from the first number, it increments by at least \(1 + 2 + \cdots + n\), since the increments at each step in the path are distinct; therefore equality must hold and thus the path from the top ends at \(N = 1 + 2 + \cdots + n\) with all the numbers \(\{1, 2, \ldots, n\}\) being close by. Let \(B\) be that position.

Consider the two left/right neighbors \(X\) and \(Y\) of the endpoint \(B\). Assume that \(B\) is to the right of the midpoint of the bottom side, and complete the equilateral triangle as shown to an apex \(C\). Consider the lightning strike from \(C\) hitting the bottom at \(D\). It travels at least \(|n/2 - 1|\) steps, by construction. But the increases must be at least \(n + 1, n + 2, \ldots\) since \(1, 2, \ldots, n\) are close to the \(A \rightarrow B\) lightning path. Then the number at \(D\) is at least

\[
(n + 1) + (n + 2) + \cdots + (n + (\lfloor n/2 - 1 \rfloor)) > 1 + 2 + \cdots + n
\]

for \(n \geq 2018\), contradiction.
§4 IMO 2018/4, proposed by Armenia

A site is any point \((x, y)\) in the plane for which \(x, y \in \{1, \ldots, 20\}\). Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones on unoccupied sites, with Amy going first; Amy has the additional restriction that no two of her stones may be at a distance equal to \(\sqrt{5}\). They stop once either player cannot move. Find the greatest \(K\) such that Amy can ensure that she places at least \(K\) stones.

The answer is \(K = 100\).

First, we show Amy can always place at least 100 stones. Indeed, treat the problem as a grid with checkerboard coloring. Then Amy can choose to always play on one of the 200 black squares. In this way, she can guarantee half the black squares, i.e., she can get \(\frac{1}{2} \cdot 200 = 100\) stones.

Second, we show Ben can prevent Amy from placing more than 100 stones. Divide into several \(4 \times 4\) squares and then further partition each \(4 \times 4\) squares as shown in the grid below.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
3 & 4 & 1 & 2 \\
2 & 1 & 4 & 3 \\
4 & 3 & 2 & 1 \\
\end{bmatrix}
\]

The squares with each label form 4-cycles by knight jumps. For each such cycle, whenever Amy plays in the cycle, Ben plays in the opposite point of the cycle, preventing Amy from playing any more stones in that original cycle. Hence Amy can play at most in \(1/4\) of the stones, as desired.
IMO 2018/5, proposed by Mongolia

Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Suppose that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer for all but finitely many positive integers $n$. Prove that $(a_n)$ is eventually constant.

The condition implies that the difference

$$S(n) = \frac{a_{n+1} - a_n}{a_1} + \frac{a_n}{a_{n+1}}$$

is an integer for all $n > N$. We proceed by $p$-adic valuation only henceforth; fix a prime $p$. Then analyzing the $\nu_p$, we immediately get that for $n > N$:

- If $\nu_p(a_n) < \nu_p(a_{n+1})$, then $\nu_p(a_{n+1}) = \nu_p(a_1)$.
- If $\nu_p(a_n) = \nu_p(a_{n+1})$, no conclusion.
- If $\nu_p(a_n) > \nu_p(a_{n+1})$, then $\nu_p(a_{n+1}) \geq \nu_p(a_1)$.

In other words:

 CLAIM — Let $p$ be a prime. Consider the sequence $\nu_p(a_{N+1}), \nu_p(a_{N+2}), \ldots$. Then either:

- We have $\nu_p(a_{N+1}) \geq \nu_p(a_{N+2}) \geq \cdots$ and so on, i.e. the sequence is weakly decreasing immediately; or
- For some index $K > N$ we have $\nu_p(a_K) < \nu_p(a_{K+1}) = \nu_p(a_{K+2}) = \cdots = \nu_p(a_1)$, i.e. the sequence “jumps” to $\nu_p(a_1)$ at some point and then stays there forever after. Note this requires $\nu_p(a_1) > 0$.

As only finitely many primes $p$ divide $a_1$, after some time $\nu_p(a_n)$ is fixed for all such $p \mid a_1$. Afterwards, the sequence satisfies $a_{n+1} \mid a_n$ for each $n$, and thus must be eventually constant.

**Remark.** This solution is almost completely $p$-adic, in the sense that I think a similar result holds if one replaces $a_n \in \mathbb{Z}$ by $a_n \in \mathbb{Z}_p$ for any particular prime $p$. In other words, the primes almost do not talk to each other.

There is one caveat: if $x_n$ is an integer sequence such that $\nu_p(x_n)$ is eventually constant for each prime then $x_n$ may not be constant. For example, take $x_n$ to be the $n$th prime! That’s why in the first claim (applied to co-finitely many of the primes), we need the stronger non-decreasing condition, rather than just eventually constant.
§6 IMO 2018/6, proposed by Poland

A convex quadrilateral \(ABCD\) satisfies \(AB \cdot CD = BC \cdot DA\). Point \(X\) lies inside \(ABCD\) so that
\[
\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.
\]
Prove that \(\angle BXA + \angle DXC = 180^\circ\).

We present two solutions by inversion. The first is the official one. The second is a solution via inversion, completed by USA5 Michael Ren.

**Official solution by inversion**  In what follows a convex quadrilateral is called *quasi-harmonic* if \(AB \cdot CD = BC \cdot DA\).

**Claim** — A quasi-harmonic quadrilateral is determined up to similarity by its angles.

(This could be expected by degrees of freedom; a quadrilateral has four degrees of freedom up to similarity; the pseudo-harmonic condition is one while the angles provide three conditions.)

**Proof.** Do some inequalities. \(\square\)

Performing an inversion at \(X\), one obtains a second quasi-harmonic quadrilateral \(A^*B^*C^*D^*\) which has the same angles as the original one, \(\angle D^* = \angle A\), \(\angle A^* = \angle B\), and so on. Thus by the claim we obtain similarity
\[
D^*A^*B^*C^* \sim ABCD.
\]
If one then maps \(D^*A^*B^*C^*\), onto \(ABCD\), the image of \(X^*\) becomes a point isogonally conjugate to \(X\). In other words, \(X\) has an isogonal conjugate in \(ABCD\).

It is well-known that this is equivalent to \(\angle BXA + \angle DXC = 180^\circ\), for example by inscribing an ellipse with foci \(X\) and \(X^*\).

**Second solution: “rhombus inversion”, by Michael Ren**  Since
\[
\frac{AB}{AD} = \frac{CB}{CD}
\]
and
\[
\frac{BA}{BC} = \frac{DA}{DC}
\]
it follows that \(B\) and \(D\) lie on an Apollonian circle \(\omega_{AC}\) through \(A\) and \(C\), while \(A\) and \(C\) lie on an Apollonian circle \(\omega_{BD}\) through \(B\) and \(D\). We let these two circles intersect at a point \(P\) inside \(ABCD\).

The main idea is then to perform an inversion about \(P\) with radius 1. We obtain:

**Lemma**
The image of \(ABCD\) is a rhombus.
Proof. By the inversion distance formula, we have
\[ \frac{1}{AB'} = \frac{PA}{AB} \cdot PB = \frac{PC}{BC} \cdot PB = \frac{1}{BC'}. \]
and so \( A'B' = B'C' \). In a similar way, we derive \( B'C' = C'D' = D'A' \), so the image is a rhombus as claimed.

Let us now translate the angle conditions. We were given that \( \angle XAB = \angle XCD \), but
\[ \angle XAB = \angle XA'P + \angle PA'B' = \angle PX'A' + \angle A'B'P \]
\[ \angle XCD = \angle CXP + \angle PCD = \angle PX'C' + \angle C'D'P \]
so subtracting these gives
\[ \angle A'X'C' = \angle A'B'P + \angle PD'C' = \angle (A'B', B'P) + \angle (PD', C'D') \]
\[ = \angle (A'B', B'P) + \angle (PD', A'B') = \angle D'PB'. \] (1)
since \( A'B' \parallel C'D' \). Similarly, we obtain
\[ \angle B'X'D' = \angle A'PC'. \] (2)

We now translate the desired condition. Since
\[ \angle AXB = \angle AXP + \angle PXB = \angle PA'X' + \angle X'B'P \]
\[ \angle CXD = \angle CXP + \angle PXD = \angle PC'X' + \angle X'D'P' \]
we compute
\[ \angle AXB + \angle CXD = (\angle PA'X' + \angle X'B'P) + (\angle PC'X' + \angle X'D'P) \]
\[ = - [ (\angle A'X'P + \angle X'PA') + (\angle PX'B' + \angle B'PX') ] \]
\[ - [ (\angle C'X'P + \angle X'PC') + (\angle PX'D' + \angle D'PX') ] \]
\[ = [ \angle PX'A' + \angle BX'P + \angle PX'C' + \angle D'X'P ] \]
\[ + [ \angle A'PX' + \angle X'PB' + \angle C'PX' + \angle X'PD' ] \]
\[ = \angle A'PB' + \angle C'PD' + \angle B'X'C + \angle D'X'A \]
and we wish to show this is equal to zero, i.e. the desired becomes
\[ \angle A'PB' + \angle C'PD' + \angle B'X'C + \angle D'X'A = 0. \] (3)

In other words, the problem is to show (1) and (2) implies (3).

Henceforth drop apostrophes. Here is the inverted diagram (with apostrophes dropped).
Let $Q$ denote the reflection of $P$ and let $Y$ denote the second intersection of $(BQC)$ and $(AQD)$. Then

$$-\angle AXC = -\angle DPB = \angle BQD = \angle BQY + \angle YQD = \angle BCY + \angle YAD$$

$$= \angle (BC, CY) + \angle (YA, AD) = \angle YCA = -\angle AYC.$$ 

Hence $XACY$ is concyclic; similarly $XBDY$ is concyclic.

**Claim —** $X \neq Y$.

**Proof.** To see this: Work pre-inversion assuming $AB < AC$. Then $Q$ was the center of $\omega_{BD}$. If $T$ was the second intersection of $BA$ with $(QBC)$, then $QB = QD = QT = \sqrt{QA \cdot QC}$, by shooting lemma. Since $\angle BAD < 180^\circ$, it follows $(QBCY)$ encloses $ABCD$ (pre-inversion). (This part is where the hypothesis that $ABCD$ is convex with $X$ inside is used.)

Finally, we do an angle chase to finish:

$$\angle DXA = \angle DXY + \angle YXA = \angle DBY + \angle YCA$$

$$= \angle (DB, YB) + \angle (CY, CA) = \angle CYB + 90^\circ$$

$$= \angle CQB + 90^\circ = -\angle APB + 90^\circ. \quad (4)$$

Similarly,

$$\angle BXC = \angle DPC + 90^\circ. \quad (5)$$

Summing (4) and (5) gives (3).

**Remark.** A difficult part of the problem in many solutions is that the conclusion is false in the directed sense, if the point $X$ is allowed to lie outside the quadrilateral. We are saved in the first solution because the equivalence of the isogonal conjugation requires $X$ inside the quadrilateral. On the other hand, in the second solution, the issue appears in the presence
of the second point $Y$. 