IMO 2012 Solution Notes

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This is an compilation of solutions for the 2012 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

2. Let $a_2, a_3, \dots, a_n$ be positive reals with product 1, where $n \ge 3$. Show that

$$(1+a_2)^2(1+a_3)^3 \dots (1+a_n)^n > n^n.$$

3. The liar's guessing game is a game played between two players A and B. The rules of the game depend on two fixed positive integers k and n which are known to both players.

   At the start of the game A chooses integers x and N with $1 \le x \le N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

   After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

   (a) If $n \ge 2^k$, then B can guarantee a win.

   (b) For all sufficiently large k, there exists an integer $n \ge (1.99)^k$ such that B cannot guarantee a win.

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a+b+c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

5. Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let $M = \overline{AL} \cap \overline{BK}$. Prove that $MK = ML$.

6. Find all positive integers n for which there exist non-negative integers $a_1, a_2, \dots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \dots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \dots + \frac{n}{3^{a_n}} = 1.$$

§1 IMO 2012/1

Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

We employ barycentric coordinates with reference $\triangle ABC$. As usual $a = BC$, $b = CA$, $c = AB$, $s = \frac{1}{2}(a+b+c)$.

It's obvious that $K = (-(s-c) : s : 0)$, $M = (0 : s-b : s-c)$. Also, $J = (-a : b : c)$. We then obtain

$$G = \left(-a : b : \frac{-as + (s-c)b}{s-b}\right).$$

It follows that

$$T = \left(0 : b : \frac{-as + (s-c)}{s-b}\right) = (0 : b(s-b) : b(s-c) - as).$$

Normalizing, we see that $T = \left(0, -\frac{b}{a}, 1 + \frac{b}{a}\right)$, from which we quickly obtain $MT = s$. Similarly, $MS = s$, so we're done.

§2 IMO 2012/2

Let $a_2, a_3, \dots, a_n$ be positive reals with product 1, where $n \ge 3$. Show that

$$(1+a_2)^2(1+a_3)^3 \dots (1+a_n)^n > n^n.$$

Try the dumbest thing possible: by AM-GM,

$$\begin{aligned}
(1+a_2)^2 &\ge 2^2 a_2 \\
(1+a_3)^3 = \left(\frac{1}{2}+\frac{1}{2}+a_3\right)^3 &\ge \frac{3^3}{2^2} a_3 \\
(1+a_4)^4 = \left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+a_4\right)^3 &\ge \frac{4^4}{3^3} a_4 \\
&\vdots
\end{aligned}$$

and so on. Multiplying these all gives the result. The inequality is strict since it's not possible that $a_2 = 1$, $a_3 = \frac{1}{2}$, et cetera.

§3 IMO 2012/3

The liar's guessing game is a game played between two players A and B. The rules of the game depend on two fixed positive integers k and n which are known to both players.

At the start of the game A chooses integers x and N with $1 \le x \le N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

(a) If $n \ge 2^k$, then B can guarantee a win.

(b) For all sufficiently large k, there exists an integer $n \ge (1.99)^k$ such that B cannot guarantee a win.

Call the players Alice and Bob.

Part (a): We prove the following.

Claim — If $n \ge 2^k+1$, then in $2k+1$ questions, Bob can rule out some number in $\{1, \dots, 2^k+1\}$ form being equal to x.

Proof. First, Bob asks the question $S_0 = \{2^k+1\}$ until Alice answers "yes" or until Bob has asked $k+1$ questions. If Alice answers "no" to all of these then Bob rules out 2^k+1. So let's assume Alice just said "yes".

Now let $T = \{1, \dots, 2^k\}$. Then, he asks k-follow up questions S_1, …, S_k defined as follows:

- $S_1 = \{1, 3, 5, 7, \dots, 2^k-1\}$ consists of all numbers in T whose least significant digit in binary is 1.
- $S_2 = \{2, 3, 6, 7, \dots, 2^k-2, 2^k-1\}$ consists of all numbers in T whose second least significant digit in binary is 1.
- More generally S_i consists of all numbers in T whose ith least significant digit in binary is 1.

WLOG Alice answers these all as "yes" (the other cases are similar). Among the last $k+1$ answers at least one must be truthful, and the number 2^k (having zeros in all relevant digits) does not appear in any of S_0, …, S_k and is ruled out. □

Thus in this way Bob can repeatedly find non-possibilities for x (and then relabel the remaining candidates 1, …, $N-1$) until he arrives at a set of at most 2^k numbers.

Part (b): It suffices to consider $n = \lceil 1.99^k \rceil$ and $N = n+1$ for large k. At the tth step, Bob asks some question S_t; we phrase each of Alice's answers in the form "$x \notin B_t$", where B_t is either S_t or its complement. (You may think of these as "bad sets"; the idea is to show we can avoid having any number appear in $k+1$ consecutive bad sets, preventing Bob from ruling out any numbers.)

Main idea: for every number $1 \le x \le N$, at time step t we define its *weight* to be

$$w(x) = 1.998^e$$

where e is the largest number such that $x \in B_{t-1} \cap B_{t-2} \cap \dots \cap B_{t-e}$.

Claim — Alice can ensure the total weight never exceeds 1.998^{k+1} for large k.

Proof. Let W_t denote the sum of weights after the th question We have $W_0 = N < 1000n$. We will prove inductively that $W_t < 1000n$ always.

At time t, Bob specifies a question S_t. We have Alice choose B_t as whichever of S_t or $\overline{S_t}$ has lesser total weight, hence at most $W_t/2$. The weights of for B_t increase by a factor of 1.998, while the weights for $\overline{B_t}$ all reset to 1. So the new total weight after time t is

$$W_{t+1} \le 1.998 \cdot \frac{W_t}{2} + \#\overline{B_t} \le 0.999W_t + n.$$

Thus if $W_t < 1000n$ then $W_{t+1} < 1000n$.

To finish, note that $1000n < 1000\left(1.99^k + 1\right) < 1.998^{k+1}$ for k large. $\square$

In particular, no individual number can have weight 1.998^{k+1}. Thus for every time step t we have

$$B_t \cap B_{t+1} \cap \dots \cap B_{t+k} = \varnothing.$$

Then once Bob stops, if he declares a set of n positive integers, and x is an integer Bob did not choose, then Alice's question history is consistent with x being Alice's number, as among any $k+1$ consecutive answers she claimed that $x \in \overline{B_t}$ for some t in that range.

Remark (Motivation)**.** In our B_t setup, let's think backwards. The problem is equivalent to avoiding $e = k + 1$ at any time step t, for any number x. That means

- have at most two elements with $e = k$ at time $t - 1$,
- thus have at most four elements with $e = k - 1$ at time $t - 2$,
- thus have at most eight elements with $e = k - 2$ at time $t - 3$,
- and so on.

We already exploited this in solving part (a). In any case it's now natural to try letting $w(x) = 2^e$, so that all the cases above sum to "equally bad" situations: since $8 \cdot 2^{k-2} = 4 \cdot 2^{k-1} = 2 \cdot 2^k$, say.

However, we then get $W_{t+1} \le \frac{1}{2}(2W_t) + n$, which can increase without bound due to contributions from numbers resetting to zero. The way to fix this is to change the weight to $w(x) = 1.998^e$, taking advantage of the little extra space we have due to having $n \ge 1.99^k$ rather than $n \ge 2^k$.

§4 IMO 2012/4

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

Answer: for arbitrary $k \in \mathbb{Z}$, we have

(i) $f(x) = kx^2$,

(ii) $f(x) = 0$ for even x, and $f(x) = k$ for odd x, and

(iii) $f(x) = 0$ for $x \equiv 0 \pmod 4$, $f(x) = k$ for odd x, and $f(x) = 4k$ for $x \equiv 2 \pmod 4$.

These can be painfully seen to work. (It's more natural to think of these as $f(x) = x^2$, $f(x) = x^2 \pmod 4$, $f(x) = x^2 \pmod 8$, and multiples thereof.)

Set $a = b = c = 0$ to get $f(0) = 0$. Then set $c = 0$ to get $f(a) = -f(-a)$, so f is even. Now

$$f(a)^2 + f(b)^2 + f(a+b)^2 = 2f(a+b)\left(f(a) + f(b)\right) + 2f(a)f(b)$$

or

$$\left(f(a+b) - (f(a) + f(b))\right)^2 = 4f(a)f(b).$$

Hence $f(a)f(b)$ is a perfect square for all $a, b \in \mathbb{Z}$. So there exists a k such that $f(n) = kg(n)^2$, where $g(n) \ge 0$. From here we recover

$$\boxed{g(a+b) = \pm g(a) \pm g(b)}.$$

Also $g(0) = 0$.

Let $k = g(1) \neq 0$. We now split into cases on $g(2)$:

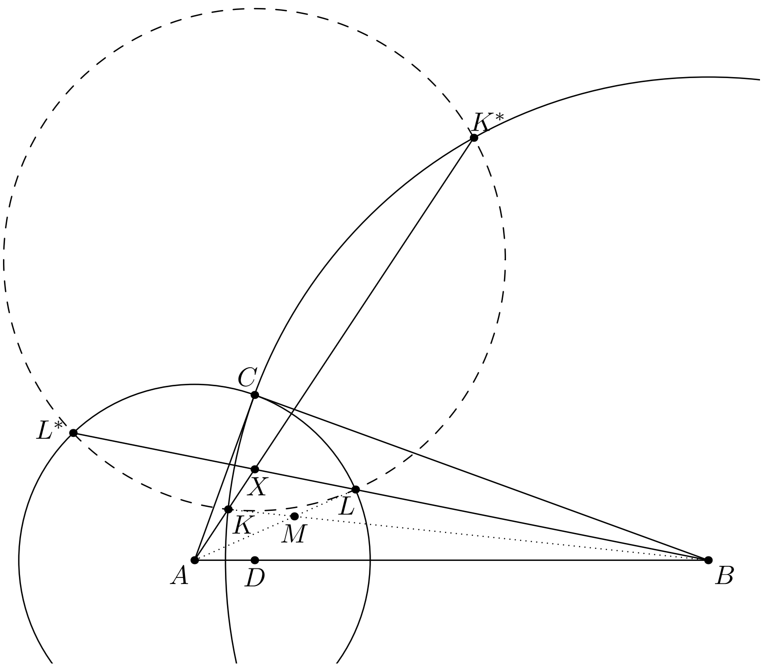
- $g(2) = 0$. Put $b = 2$ in original to get $g(a+2) = \pm g(a) = +g(a)$.
- $g(2) = 2c$. Cases on $g(4)$:
  - $g(4) = 0$, then we get $(gn)_{n \ge 0} = (0, 1, 2, 1, 0, 1, 2, 1, \dots)$. This works.
  - $g(4) = 2k$. No good, as $g(4) = \pm g(2) \pm g(2)$.
  - $g(4) = 4k$. This only happens when $g(1) = k$, $g(2) = 2k$, $g(3) = 3k$, $g(4) = 4k$. Then
    * $g(5) = \pm 3k \pm 2k = \pm 4k \pm k$.
    * $g(6) = \pm 4k \pm 2k = \pm 5k \pm k$.
    * $\dots$

    and so by induction $g(n) = nc$.

§5 IMO 2012/5

Let ABC be a triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let $M = \overline{AL} \cap \overline{BK}$. Prove that $MK = ML$.

Let ω_A and ω_B be the circles through C centered at A and B; extend rays AK and BL to hit ω_B and ω_A again at K^*, L^*. By radical center X, we have KLK^*L^* is cyclic, say with circumcircle ω.



By orthogonality of (A) and (B) we find that $\overline{AL}$, $\overline{AL^*}$, $\overline{BK}$, $\overline{BK^*}$ are tangents to ω (in particular, KLK^*L^* is harmonic). In particular $\overline{MK}$ and $\overline{ML}$ are tangents to ω, so $MK = ML$.

§6 IMO 2012/6, proposed by Dusan Djukic (SRB)

Find all positive integers n for which there exist non-negative integers $a_1, a_2, \dots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \dots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \dots + \frac{n}{3^{a_n}} = 1.$$

The answer is $n \equiv 0, 1 \pmod 4$. To see these are necessary, note that taking the latter equation modulo 2 gives

$$1 = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \dots + \frac{n}{3^{a_n}} \equiv 1 + 2 + .. + n \pmod 2.$$

Now we prove these are sufficient. The following nice construction was posted on AOPS by the user `cfheolpiixn`.

Claim — If $n = 2k - 1$ works then so does $n = 2k$.

Proof. Replace

$$\frac{k}{3^r} = \frac{k}{3^{r+1}} + \frac{2k}{3^{r+1}}. \qquad (*)$$

□

Claim — If $n = 4k + 2$ works then so does $n = 4k + 13$.

Proof. First use the identity

$$\frac{k+2}{3^r} = \frac{k+2}{3^{r+2}} + \frac{4k+3}{3^{r+3}} + \frac{4k+5}{3^{r+3}} + \frac{4k+7}{3^{r+3}} + \frac{4k+9}{3^{r+3}} + \frac{4k+11}{3^{r+3}} + \frac{4k+13}{3^{r+3}}$$

to fill in the odd numbers. The even numbers can then be instantiated with $(*)$ too. □

Thus it suffices to construct base cases for $n = 1$, $n = 5$, $n = 9$. They are

$$\begin{aligned} 1 &= \frac{1}{3^1} + \frac{2}{3^1} \\ &= \frac{1}{3^2} + \frac{2}{3^2} + \frac{3}{3^2} + \frac{4}{3^3} + \frac{5}{3^3} \\ &= \frac{1}{3^2} + \frac{2}{3^3} + \frac{3}{3^3} + \frac{4}{3^3} + \frac{5}{3^3} + \frac{6}{3^4} + \frac{7}{3^4} + \frac{8}{3^4} + \frac{9}{3^4}. \end{aligned}$$