This is a compilation of solutions for the 2010 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums. Corrections and comments are welcome!
§0 Problems

1. Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that for all \( x, y \in \mathbb{R} \),
   \[ f([x \mid y] = f(x) \lfloor f(y) \rfloor. \]

2. Let \( I \) be the incenter of a triangle \( ABC \) and let \( \Gamma \) be its circumcircle. Let line \( AI \)
   intersect \( \Gamma \) again at \( D \). Let \( E \) be a point on arc \( BDC \) and \( F \) a point on side \( BC \)
   such that
   \[ \angle BAF = \angle CAE < \frac{1}{2} \angle BAC. \]
   Finally, let \( G \) be the midpoint of \( IF \). Prove that \( DG \) and \( EI \) intersect on \( \Gamma \).

3. Find all functions \( g : \mathbb{Z}_{> 0} \to \mathbb{Z}_{> 0} \) such that
   \[ (g(m) + n)(g(n) + m) \]
   is always a perfect square.

4. Let \( P \) be a point interior to triangle \( ABC \) (with \( CA \neq CB \)). The lines \( AP, BP \)
   and \( CP \) meet again its circumcircle \( \Gamma \) at \( K, L, M \), respectively. The tangent line
   at \( C \) to \( \Gamma \) meets the line \( AB \) at \( S \). Show that from \( SC = SP \) follows \( MK = ML \).

5. Each of the six boxes \( B_1, B_2, B_3, B_4, B_5, B_6 \) initially contains one coin. The
   following two types of operations are allowed:
   a) Choose a non-empty box \( B_j, 1 \leq j \leq 5 \), remove one coin from \( B_j \) and add
      two coins to \( B_{j+1} \);
   b) Choose a non-empty box \( B_k, 1 \leq k \leq 4 \), remove one coin from \( B_k \) and swap
      the contents (possibly empty) of the boxes \( B_{k+1} \) and \( B_{k+2} \).
   Determine if there exists a finite sequence of operations of the allowed types, such
   that the five boxes \( B_1, B_2, B_3, B_4, B_5 \) become empty, while box \( B_6 \) contains exactly
   \( 2010^2010^2010 \) coins.

6. Let \( a_1, a_2, a_3, \ldots \) be a sequence of positive real numbers, and \( s \) be a positive integer,
   such that
   \[ a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\} \text{ for all } n > s. \]
   Prove there exist positive integers \( \ell \leq s \) and \( N \), such that
   \[ a_n = a_\ell + a_{n-\ell} \text{ for all } n \geq N. \]
§1 IMO 2010/1

Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f([x]y) = f(x) \lfloor f(y) \rfloor.$$ 

The only solutions are $f(x) \equiv c$, where $c = 0$ or $1 \leq c < 2$. It’s easy to see these work. Plug in $x = 0$ to get $f(0) = f(0) \lfloor f(y) \rfloor$, so either

$$1 \leq f(y) < 2 \quad \forall y \quad \text{or} \quad f(0) = 0$$

In the first situation, plug in $y = 0$ to get $f(x) \lfloor f(0) \rfloor = f(0)$, thus $f$ is constant. Thus assume henceforth $f(0) = 0$.

Now set $x = y = 1$ to get

$$f(1) = f(1) \lfloor f(1) \rfloor$$

so either $f(1) = 0$ or $1 \leq f(1) < 2$. We split into cases:

- If $f(1) = 0$, pick $x = 1$ to get $f(y) \equiv 0$.
- If $1 \leq f(1) < 2$, then $y = 1$ gives

$$f([x]) = f(x)$$

from $y = 1$, in particular $f(x) = 0$ for $0 \leq x < 1$. Choose $(x, y) = (2, \frac{1}{2})$ to get $f(1) = f(2) \lfloor f \left( \frac{1}{2} \right) \rfloor = 0$. 


§2 IMO 2010/2

Let $I$ be the incenter of a triangle $ABC$ and let $\Gamma$ be its circumcircle. Let line $AI$ intersect $\Gamma$ again at $D$. Let $E$ be a point on arc $BDC$ and $F$ a point on side $BC$ such that

$$\angle BAF = \angle CAE < \frac{1}{2} \angle BAC.$$

Finally, let $G$ be the midpoint of $IF$. Prove that $DG$ and $EI$ intersect on $\Gamma$.

Let $EI$ meet $\Gamma$ again at $K$. Then it suffices to show that $KD$ bisects $IF$. Let $AF$ meet $\Gamma$ again at $H$, so $HE \parallel BC$. By Pascal theorem on $AHEKD$ we then obtain that $P = AH \cap KD$ lies on a line through $I$ parallel to $BC$.

Let $I_A$ be the $A$-excenter, and set $Q = I_AF \cap IP$, and $T = AIDI_A \cap BFC$. Then

$$-1 = (AD; TI_A) \equiv (IQ; \infty P)$$

where $\infty$ is the point at infinity along $IPQ$. Thus $P$ is the midpoint of $IQ$. Since $D$ is the midpoint of $II_A$ by “Fact 5”, it follows that $DP$ bisects $IF$.
§3 IMO 2010/3, proposed by Gabriel Carroll (USA)

Find all functions $g : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ such that

$$(g(m) + n)(g(n) + m)$$

is always a perfect square.

For $c \geq 0$, the function $g(n) = n + c$ works; we prove this is the only possibility.

First, the main point of the problem is that

**Claim** — We have $g(n) \equiv g(n') \pmod{p} \implies n \equiv n' \pmod{p}$.

**Proof.** Pick a large integer $M$ such that

$$\nu_p(M + g(n)), \quad \nu_p(M + g(n'))$$

are both odd. (It’s not hard to see this is always possible.) Now, since each of

$$(M + g(n))(n + g(M))$$

$$((M + g(n'))(n' + g(M))$$

is a square, we get $g(n) \equiv g(n') \equiv -M \pmod{p}$. \hfill \Box$

This claim implies that

- The numbers $g(n)$ and $g(n + 1)$ differ by $\pm 1$ for any $n$, and
- The function $g$ is injective.

It follows $g$ is a linear function with slope $\pm 1$, hence done.
§4 IMO 2010/4

Let \( P \) be a point interior to triangle \( ABC \) (with \( CA \neq CB \)). The lines \( AP, BP \) and \( CP \) meet again its circumcircle \( \Gamma \) at \( K, L, M \), respectively. The tangent line at \( C \) to \( \Gamma \) meets the line \( AB \) at \( S \). Show that from \( SC = SP \) follows \( MK = ML \).

We present two solutions using harmonic bundles.

**First solution (Evan Chen)** Let \( N \) be the antipode of \( M \), and let \( NP \) meet \( \Gamma \) again at \( D \). Focus only on \( CDMN \) for now (ignoring the condition). Then \( C \) and \( D \) are feet of altitudes in \( \triangle MNP \); it is well-known that the circumcircle of \( \triangle CDP \) is orthogonal to \( \Gamma \) (passing through the orthocenter of \( \triangle MPN \)).

![Harmonic Bundles Diagram](image)

Now, we are given that point \( S \) is such that \( SC \) is tangent to \( \Gamma \), and \( SC = SP \). It follows that \( S \) is the circumcenter of \( \triangle CDP \), and hence \( SC \) and \( SD \) are tangents to \( \Gamma \).

Then \(-1 = (AB; CD) = (KL; MN) \). Since \( MN \) is a diameter, this implies \( MK = ML \).

**Remark.** I think it’s more natural to come up with this solution in reverse. Namely, suppose we define the points the other way: let \( SD \) be the other tangent, so \((AB; CD) = -1 \). Then project through \( P \) to get \((KL; MN) = -1 \), where \( N \) is the second intersection of \( DP \). However, if \( ML = MK \) then \( KMLN \) must be a kite. Thus one can recover the solution in reverse.

**Second solution (Sebastian Jeon)** We have

\[ SP^2 = SC^2 = SA \cdot SB \implies \angle SPA = \angle PBA = \angle LBA = \angle LKA = \angle LKP \]

(the latter half is Reim’s theorem). Therefore \( SP \) and \( LK \) are parallel.

Now, let \( SP \) meet \( \Gamma \) again at \( X \) and \( Y \), and let \( Q \) be the antipode of \( P \) on \( \Gamma \). Then

\[ SP^2 = SQ^2 = SX \cdot SY \implies (PQ; XY) = -1 \implies \angle QCP = 90^\circ \]

that \( CP \) bisects \( \angle XCY \). Since \( XY \parallel KL \), it follows \( CP \) bisects to \( \angle LCK \) too.
§5 IMO 2010/5, proposed by Netherlands

Each of the six boxes $B_1$, $B_2$, $B_3$, $B_4$, $B_5$, $B_6$ initially contains one coin. The following two types of operations are allowed:

1. Choose a non-empty box $B_j$, $1 \leq j \leq 5$, remove one coin from $B_j$ and add two coins to $B_{j+1}$;
2. Choose a non-empty box $B_k$, $1 \leq k \leq 4$, remove one coin from $B_k$ and swap the contents (possibly empty) of the boxes $B_{k+1}$ and $B_{k+2}$.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes $B_1$, $B_2$, $B_3$, $B_4$, $B_5$ become empty, while box $B_6$ contains exactly $2010^{2010^{2010}}$ coins.

First,

$$(1,1,1,1,1,1) \rightarrow (0,3,1,0,3,1) \rightarrow (0,0,7,0,0,7) \rightarrow (0,0,0,6,1,2,7) \rightarrow (0,0,6,1,0,11) \rightarrow (0,0,6,0,11,0) \rightarrow (0,0,5,11,0,0).$$

and henceforth we ignore boxes $B_1$ and $B_2$, looking at just the last four boxes; so we write the current position as $(5,11,0,0)$.

We prove a lemma:

**Claim** — Let $k \geq 0$ and $n > 0$. From $(k,n,0,0)$ we may reach $(k-1,2^n,0,0)$.

**Proof.** Working with only the last three boxes for now,

$$(n,0,0) \rightarrow (n-1,2,0) \rightarrow (n-1,0,4) \rightarrow (n-2,4,0) \rightarrow (n-2,0,8) \rightarrow (n-3,8,0) \rightarrow (n-3,0,16) \rightarrow \cdots \rightarrow (1,2^{n-1},0) \rightarrow (1,0,2^n) \rightarrow (0,2^n,0).$$

Finally we have $(k,n,0,0) \rightarrow (k,0,2^n,0) \rightarrow (k-1,2^n,0,0)$. \(\square\)

Now from $(5,11,0,0)$ we go as follows:

$$(5,11,0,0) \rightarrow (4,2^{11},0,0) \rightarrow \left(3,2^{211},0,0\right) \rightarrow \left(2,2^{2311},0,0\right) \rightarrow \left(1,2^{22211},0,0\right) \rightarrow \left(0,2^{222211},0,0\right).$$

Let $A = 2^{222211} > 2010^{2010^{2010}} = B$. Then by using move 2 repeatedly on the fourth box (i.e., throwing away several coins by swapping the empty $B_5$ and $B_6$), we go from $(0,A,0,0)$ to $(0,B/4,0,0)$. From there we reach $(0,0,0,B)$. 


§6 IMO 2010/6

Let \( a_1, a_2, a_3, \ldots \) be a sequence of positive real numbers, and \( s \) be a positive integer, such that
\[
a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\}
\]
for all \( n > s \).

Prove there exist positive integers \( \ell \leq s \) and \( N \), such that
\[
a_n = a_\ell + a_{n-\ell} \quad \text{for all} \quad n \geq N.
\]

Let
\[
w_1 = \frac{a_1}{1}, \quad w_2 = \frac{a_2}{2}, \quad \ldots, \quad w_s = \frac{a_s}{s}.
\]
(The choice of the letter \( w \) is for “weight”.) We claim the right choice of \( \ell \) is the one maximizing \( w_\ell \).

Our plan is to view each \( a_n \) as a linear combination of the weights \( w_1, \ldots, w_s \) and track their coefficients.

To this end, let’s define an \( n \)-type to be a vector \( T = \langle t_1, \ldots, t_s \rangle \) of nonnegative integers such that
- \( n = t_1 + \cdots + t_s \); and
- \( t_i \) is divisible by \( i \) for every \( i \).

We then define its valuation as \( v(T) = \sum_{i=1}^{s} w_i t_i \).

Now we define a \( n \)-type to be valid according to the following recursive rule. For \( 1 \leq n \leq s \) the only valid \( n \)-types are
\[
T_1 = \langle 1, 0, 0, \ldots, 0 \rangle \\
T_2 = \langle 0, 2, 0, \ldots, 0 \rangle \\
T_3 = \langle 0, 0, 3, \ldots, 0 \rangle \\
\vdots \\
T_s = \langle 0, 0, 0, \ldots, s \rangle
\]
for \( n = 1, \ldots, s \), respectively. Then for any \( n > s \), an \( n \)-type is valid if it can be written as the sum of a valid \( k \)-type and a valid \( (n-k) \)-type, componentwise. These represent the linear combinations possible in the recursion; in other words the recursion in the problem is phrased as
\[
a_n = \max_{T \text{ is a valid } n \text{-type}} v(T).
\]

In fact, we have the following description of valid \( n \)-types:

**Claim** — Assume \( n > s \). Then an \( n \)-type \( \langle t_1, \ldots, t_s \rangle \) is valid if and only if either
- there exist indices \( i < j \) with \( i + j > s \), \( t_i \geq i \) and \( t_j \geq j \); or
- there exists an index \( i > s/2 \) with \( t_i \geq 2i \).

**Proof.** Immediate by forwards induction on \( n > s \) that all \( n \)-types have this property.

The reverse direction is by downwards induction on \( n \). Indeed if \( \sum_{i=1}^{s} \frac{t_i}{i} > 2 \), then we may subtract off on of \( \{T_1, \ldots, T_s\} \) while preserving the condition; and the case \( \sum_{i=1}^{s} \frac{t_i}{i} = 2 \) is essentially by definition.
Remark. The claim is a bit confusingly stated in its two cases; really the latter case should be thought of as the situation $i = j$ but requiring that $t_i/i$ is counted with multiplicity.

Now, for each $n > s$ we pick a valid $n$-type $T_n$ with $a_n = v(T_n)$; if there are ties, we pick one for which the $\ell$th entry is as large as possible.

**Claim** — For any $n > s$ and index $i \neq \ell$, the $i$th entry of $T_n$ is at most $2s + \ell i$.

**Proof.** If not, we can go back $i\ell$ steps to get a valid $(n - i\ell)$-type $T$ achieved by decreasing the $i$th entry of $T_n$ by $i\ell$. But then we can add $\ell$ to the $\ell$th entry $i$ times to get another $n$-type $T'$ which obviously has valuation at least as large, but with larger $\ell$th entry. \(\square\)

Now since all other entries in $T_n$ are bounded, eventually the sequence $(T_n)_{n > s}$ just consists of repeatedly adding 1 to the $\ell$th entry, as required.

Remark. One big step is to consider $w_k = a_k/k$. You can get this using wishful thinking or by examining small cases. (In addition this normalization makes it easier to see why the largest $w$ plays an important role, since then in the definition of type, the $n$-types all have a sum of $n$. Unfortunately, it makes the characterization of valid $n$-types somewhat clumsier too.)