IMO 2008 Solution Notes

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This is a compilation of solutions for the 2008 IMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, C_2 are concyclic.

2. Let x, y, z be real numbers with $xyz = 1$, all different from 1. Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

and show that equality holds for infinitely many choices of rational numbers x, y, z.

3. Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime factor greater than $2n + \sqrt{2n}$.

4. Find all functions f from the positive reals to the positive reals such that

$$\frac{f(w)^2 + f(x)^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z satisfying $wx = yz$.

5. Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. There are $2n$ lamps labelled 1, 2, …, $2n$ each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on). Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off. Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on. Determine $\frac{N}{M}$.

6. Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

§1 Solutions to Day 1

§1.1 IMO 2008/1, proposed by Andrey Gavrilyuk (RUS)

Available online at https://aops.com/community/p1190553.

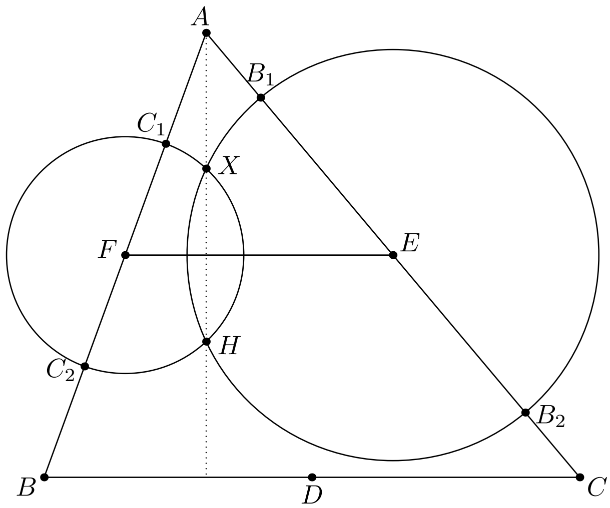
Problem statement

Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, C_2 are concyclic.

We show two solutions.

¶ First solution using power of a point. Let D, E, F be the centers of Γ_A, Γ_B, Γ_C (in other words, the midpoints of the sides).

We first show that B_1, B_2, C_1, C_2 are concyclic. It suffices to prove that A lies on the radical axis of the circles Γ_B and Γ_C.



Let X be the second intersection of Γ_B and Γ_C. Clearly $\overline{XH}$ is perpendicular to the line joining the centers of the circles, namely $\overline{EF}$. But $\overline{EF} \parallel \overline{BC}$, so $\overline{XH} \perp \overline{BC}$. Since $\overline{AH} \perp \overline{BC}$ as well, we find that A, X, H are collinear, as needed.

Thus, B_1, B_2, C_1, C_2 are concyclic. Similarly, C_1, C_2, A_1, A_2 are concyclic, as are A_1, A_2, B_1, B_2. Now if any two of these three circles coincide, we are done; else the pairwise radical axii are not concurrent, contradiction. (Alternatively, one can argue directly that O is the center of all three circles, by taking the perpendicular bisectors.)

¶ Second solution using length chase (Ritwin Narra). We claim the circumcenter O of $\triangle ABC$ is in fact the center of $(A_1A_2B_1B_2C_1C_2)$.

Define D, E, F as before. Then since $\overline{OD} \perp \overline{A_1A_2}$ and $DA_1 = DA_2$, which means $OA_1 = OA_2$. Similarly, we have $OB_1 = OB_2$ and $OC_1 = OC_2$.

Now since $DA_1 = DA_2 = DH$, we have $OA_1^2 = OD^2 + HD^2$. We seek to show

$$OD^2 + HD^2 = OE^2 + HE^2 = OF^2 + HF^2.$$

This is clear by Appollonius's Theorem since D, E, and F lie on the nine-point circle, which is centered at the midpoint of $\overline{OH}$.

§1.2 IMO 2008/2, proposed by Walther Janous (AUT)

Available online at https://aops.com/community/p1190551.

Problem statement

Let x, y, z be real numbers with $xyz = 1$, all different from 1. Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \ge 1$$

and show that equality holds for infinitely many choices of rational numbers x, y, z.

Let $x = a/b$, $y = b/c$, $z = c/a$, so we want to show

$$\left(\frac{a}{a-b}\right)^2 + \left(\frac{b}{b-c}\right)^2 + \left(\frac{c}{c-a}\right)^2 \ge 1.$$

A very boring computation shows this is equivalent to

$$\frac{(a^2b + b^2c + c^2a - 3abc)^2}{(a-b)^2(b-c)^2(c-a)^2} \ge 0$$

which proves the inequality (and it is unsurprising we are in such a situation, given that there is an infinite curve of rationals).

For equality, it suffices to show there are infinitely many integer solutions to

$$a^2b + b^2c + c^2a = 3abc \iff \frac{a}{c} + \frac{b}{a} + \frac{c}{a} = 3$$

or equivalently that there are infinitely many rational solutions to

$$u + v + \frac{1}{uv} = 3.$$

For any $0 \ne u \in \mathbb{Q}$ the real solution for u is

$$v = \frac{-u + (u-1)\sqrt{1 - 4/u} + 3}{2}$$

and there are certainly infinitely many rational numbers u for which $1 - 4/u$ is a rational square (say, $u = \frac{-4}{q^2-1}$ for $q \ne \pm 1$ a rational number).

§1.3 IMO 2008/3, proposed by Kęstutis Česnavičius (LTU)

Available online at https://aops.com/community/p1190546.

Problem statement

Prove that there are infinitely many positive integers n such that n^2+1 has a prime factor greater than $2n+\sqrt{2n}$.

The idea is to pick the prime p first!

Select any large prime $p \ge 2013$, and let $h = \left\lceil \sqrt{p} \right\rceil$. We will try to find an n such that

$$n \le \frac{1}{2}(p-h) \quad \text{and} \quad p \mid n^2+1.$$

This implies $p \ge 2n+\sqrt{p}$ which is enough to ensure $p \ge 2n+\sqrt{2n}$.

Assume $p \equiv 1 \pmod 8$ henceforth. Then there exists some $\frac{1}{2}p < x < p$ such that $x^2 \equiv -1 \pmod p$, and we set

$$x = \frac{p+1}{2} + t.$$

Claim — We have $t \ge \frac{h-1}{2}$ and hence may take $n = p - x$.

Proof. Assume for contradiction this is false; then

$$\begin{aligned} 0 &\equiv 4(x^2+1) \pmod p \\ &= (p+1+2t)^2 + 4 \\ &\equiv (2t+1)^2 + 4 \pmod p \\ &< h^2 + 4 \end{aligned}$$

So we have that $(2t+1)^2+4$ is positive and divisible by p, yet at most $\left\lceil \sqrt{p} \right\rceil^2 + 4 < 2p$. So it must be the case that $(2t+1)^2+4 = p$, but this has no solutions modulo 8. $\square$

§2 Solutions to Day 2

§2.1 IMO 2008/4, proposed by Hojoo Lee (KOR)

Available online at https://aops.com/community/p1191683.

Problem statement

Find all functions f from the positive reals to the positive reals such that

$$\frac{f(w)^2+f(x)^2}{f(y^2)+f(z^2)} = \frac{w^2+x^2}{y^2+z^2}$$

for all positive real numbers w, x, y, z satisfying $wx=yz$.

The answers are $f(x) \equiv x$ and $f(x) \equiv 1/x$. These work, so we show they are the only ones.

First, setting (t,t,t,t) gives $f(t^2)=f(t)^2$. In particular, $f(1)=1$. Next, setting $(t,1,\sqrt{t},\sqrt{t})$ gives

$$\frac{f(t)^2+1}{2f(t)} = \frac{t^2+1}{2t}$$

which as a quadratic implies $f(t) \in \{t, 1/t\}$.

Now assume $f(a)=a$ and $f(b)=1/b$. Setting $(\sqrt{a},\sqrt{b},1,\sqrt{ab})$ gives

$$\frac{a+1/b}{f(ab)+1} = \frac{a+b}{ab+1}.$$

One can check the two cases on $f(ab)$ each imply $a=1$ and $b=1$ respectively. Hence the only answers are those claimed.

§2.2 IMO 2008/5, proposed by Bruno Le Floch and Ilia Smilga (FRA)

Available online at https://aops.com/community/p1191679.

Problem statement

Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. There are $2n$ lamps labelled 1, 2, ..., $2n$ each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on). Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off. Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on. Determine $\frac{N}{M}$.

The answer is 2^{k-n}.

Consider the following map Ψ from N-sequences to M-sequences:

- change every instance of $n+1$ to 1;
- change every instance of $n+2$ to 2;

  ⋮

- change every instance of $2n$ to n.

(For example, suppose $k = 9$, $n = 3$; then $144225253 \mapsto 111222223$.)

Clearly this is map is well-defined and surjective. So all that remains is:

Claim — Every M-sequence has exactly 2^{k-n} pre-images under Ψ.

Proof. Indeed, suppose that there are c_1 instances of lamp 1. Then we want to pick an odd subset of the 1's to change to $n+1$'s, so 2^{c_1-1} ways to do this. And so on. Hence the number of pre-images is

$$\prod_i 2^{c_i-1} = 2^{k-n}. \qquad \square$$

§2.3 IMO 2008/6, proposed by Vladimir Shmarov (RUS)

Available online at https://aops.com/community/p1191671.

Problem statement

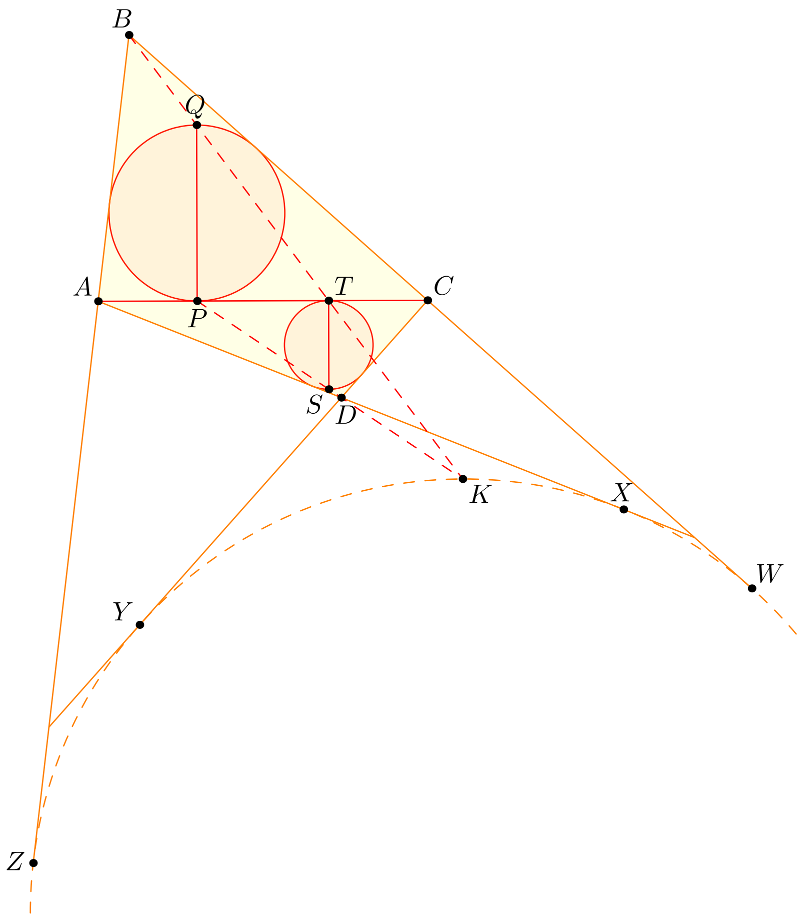
Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

By the external version of Pitot theorem, the existence of ω implies that

$$BA + AD = CB + CD.$$

Let $\overline{PQ}$ and $\overline{ST}$ be diameters of ω_1 and ω_2 with $P, T \in \overline{AC}$. Then the length relation on $ABCD$ implies that P and T are reflections about the midpoint of $\overline{AC}$.

Now orient AC horizontally and let K be the "uppermost" point of ω, as shown.



Consequently, a homothety at B maps Q, T, K to each other (since T is the uppermost of the excircle, Q of the incircle). Similarly, a homothety at D maps P, S, K to each other. As $\overline{PQ}$ and $\overline{ST}$ are parallel diameters it then follows K is the exsimilicenter of ω_1 and ω_2.