IMO 2007 Solution Notes

Compiled by Evan Chen

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This is an compilation of solutions for the 2007 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums. Corrections and comments are welcome!

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§0 Problems

1. Real numbers $a_1, a_2, \ldots, a_n$ are fixed. For each $1 \leq i \leq n$ we let $d_i = \max\{a_j : 1 \leq j \leq i\} - \min\{a_j : i \leq j \leq n\}$ and let $d = \max\{d_i : 1 \leq i \leq n\}$.

   (a) Prove that for any real numbers $x_1 \leq \cdots \leq x_n$ we have
   \[\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{1}{2} d.\]

   (b) Moreover, show that there exists some choice of $x_1 \leq \cdots \leq x_n$ which achieves equality.

2. Consider five points $A, B, C, D$ and $E$ such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let $\ell$ be a line passing through $A$. Suppose that $\ell$ intersects the interior of the segment $DC$ at $F$ and intersects line $BC$ at $G$. Suppose also that $EF = EG = EC$. Prove that $\ell$ is the bisector of angle $DAB$.

3. In a mathematical competition some competitors are (mutual) friends. Call a group of competitors a clique if each two of them are friends. Given that the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

4. In triangle $ABC$ the bisector of $\angle BCA$ meets the circumcircle again at $R$, the perpendicular bisector of $BC$ at $P$, and the perpendicular bisector of $AC$ at $Q$. The midpoint of $BC$ is $K$ and the midpoint of $AC$ is $L$. Prove that the triangles $RPK$ and $RQL$ have the same area.

5. Let $a$ and $b$ be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

6. Let $n$ be a positive integer. Consider
   \[S = \{(x, y, z) \mid x, y, z \in \{0, 1, \ldots, n\}, \quad x + y + z > 0\}\]
as a set of $(n + 1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains $S$ but does not include $(0, 0, 0)$. 
§1 IMO 2007/1

Real numbers $a_1, a_2, \ldots, a_n$ are fixed. For each $1 \leq i \leq n$ we let $d_i = \max\{a_j : 1 \leq j \leq i\} - \min\{a_j : i \leq j \leq n\}$ and let $d = \max\{d_i : 1 \leq i \leq n\}$.

(a) Prove that for any real numbers $x_1 \leq \cdots \leq x_n$ we have

$$\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{1}{2} d.$$ 

(b) Moreover, show that there exists some choice of $x_1 \leq \cdots \leq x_n$ which achieves equality.

Note that we can dispense of $d_i$ immediately by realizing that the definition of $d$ just says

$$d = \max_{1 \leq i \leq j \leq n} (a_i - a_j).$$

If $a_1 \leq \cdots \leq a_n$ are already nondecreasing then $d = 0$ and there is nothing to prove (for the equality case, just let $x_i = a_i$), so we will no longer consider this case.

Otherwise, consider any indices $i < j$ with $a_i > a_j$. We first prove (a) by applying the following claim with $p = a_i$ and $q = a_j$:

**Claim** — For any $p \leq q$, we have either $|p - a_i| \geq \frac{1}{2}(a_i - a_j)$ or $|q - a_j| \geq \frac{1}{2}(a_i - a_j)$.

**Proof.** Assume for contradiction both are false. Then $p > a_i - \frac{1}{2}(a_i - a_j) = a_j + \frac{1}{2}(a_i - a_j) > q$, contradiction.

As for (b), we let $i < j$ be any indices for which $a_i - a_j = d > 0$ achieves the maximal difference. We then define $x_\bullet$ in three steps:

- We set $x_k = \frac{a_i + a_j}{2}$ for $k = i, \ldots, j$.
- We recursively set $x_k = \max(x_{k-1}, a_k)$ for $k = j+1, j+2, \ldots$.
- We recursively set $x_k = \min(x_{k+1}, a_k)$ for $k = i-1, i-2, \ldots$.

By definition, these $x_\bullet$ are weakly increasing. To prove this satisfies (b) we only need to check that

$$|x_k - a_k| \leq \frac{a_i - a_j}{2} \quad (\ast)$$

for any index $k$ (as equality holds for $k = i$ or $k = j$).

We note $(\ast)$ holds for $i < k < j$ by construction. For $k > j$, note that $x_k \in \{a_j, a_{j+1}, \ldots, a_k\}$ by construction, so $(\ast)$ follows from our choice of $i$ and $j$ giving the largest possible difference; the case $k < i$ is similar.
§2 IMO 2007/2, proposed by Charles Leytem (LUX)

Consider five points $A, B, C, D$ and $E$ such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let $\ell$ be a line passing through $A$. Suppose that $\ell$ intersects the interior of the segment $DC$ at $F$ and intersects line $BC$ at $G$. Suppose also that $EF = EG = EC$. Prove that $\ell$ is the bisector of angle $DAB$.

Let $M, N, P$ denote the midpoints of $CF, CG, AC$ (noting $P$ is also the midpoint of $BD$).

By a homothety at $C$ with ratio $\frac{1}{2}$, we find $MNP$ is the image of line $\ell \equiv AGF$.

However, since we also have $EM \perp CF$ and $EN \perp CG$ (from $EF = EG = EC$) we conclude $PMN$ is the Simson line of $E$ with respect to $\triangle BCD$, which implies $EP \perp BD$. In other words, $EP$ is the perpendicular bisector of $BD$, so $E$ is the midpoint of arc $BDC$.

Finally,

$$\angle(AB, \ell) = \angle(CD, MNP) = \angle CMN = \angle CEN = 90^\circ - \angle NCE = 90^\circ + \angle ECB$$

which means that $\ell$ is parallel to a bisector of $\angle BCD$, and hence to one of $\angle BAD$.

(Moreover since $F$ lies on the interior of $CD$, it is actually the internal bisector)
§3 IMO 2007/3, proposed by Vasily Astakhov (RUS)

In a mathematical competition some competitors are (mutual) friends. Call a group of competitors a **clique** if each two of them are friends. Given that the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Take the obvious graph interpretation $G$. We paint red any vertices in one of the maximal cliques $K$, which we assume has $2r$ vertices, and paint the remaining vertices green. We let $\alpha(\bullet)$ denote the clique number.

Initially, let the two rooms $A = K, B = G - K$.

**Claim** — We can move at most $r$ vertices of $A$ into $B$ to arrive at $\alpha(A) \leq \alpha(B) \leq \alpha(A) + 1$.

**Proof.** This is actually obvious by discrete continuity. We move one vertex at a time, noting $\alpha(A)$ decreases by one at each step, while $\alpha(B)$ increases by either zero or one at each step.

We stop once $\alpha(B) \geq \alpha(A)$, which happens before we have moved $r$ vertices (since then we have $\alpha(B) \geq r = \alpha(A)$). The conclusion follows.

So let’s consider the situation

$$\alpha(A) = k \geq r \quad \text{and} \quad \alpha(B) = k + 1.$$

At this point $A$ is a set of $k$ red vertices, while $B$ has the remaining $2r - k$ red vertices (and all the green ones). An example is shown below with $k = 4$ and $2r = 6$.

Now, if we can move any red vertex from $B$ back to $A$ without changing the clique number of $B$, we do so, and win.

Otherwise, it must be the case that every $(k + 1)$-clique in $B$ uses every red vertex in $B$. For each $(k + 1)$-clique in $B$ (in arbitrary order), we do the following procedure.

- If all $k + 1$ vertices are still green, pick one and re-color it blue. This is possible since $k + 1 > 2r - k$.
- Otherwise, do nothing.
Then we move all the blue vertices from $B$ to $A$, one at a time, in the same order we re-colored them. This forcibly decreases the clique number of $B$ to $k$, since the clique number is $k + 1$ just before the last blue vertex is moved, and strictly less than $k + 1$ (hence equal to $k$) immediately after that.

**Claim** — After this, $\alpha(A) = k$ still holds.

**Proof.** Assume not, and we have a $(k + 1)$-clique which uses $b$ blue vertices and $(k + 1) - b$ red vertices in $A$. Together with the $2r - k$ red vertices already in $B$ we then get a clique of size

$$b + ((k + 1 - b)) + (2r - k) = 2r + 1$$

which is a contradiction. □
§4 IMO 2007/4, proposed by Marek Pechal (CZE)

In triangle $ABC$ the bisector of $\angle BCA$ meets the circumcircle again at $R$, the perpendicular bisector of $BC$ at $P$, and the perpendicular bisector of $AC$ at $Q$. The midpoint of $BC$ is $K$ and the midpoint of $AC$ is $L$. Prove that the triangles $RPK$ and $RQL$ have the same area.

We first begin by proving the following claim.

**Claim** — We have $CQ = PR$ (equivalently, $CP = QR$).

**Proof.** Let $O = \overline{LQ} \cap \overline{KP}$ be the circumcenter. Then

$$\angle OPQ = \angle KPC = 90^\circ - \angle PCK = 90^\circ - \angle LCQ = \angle CQL = \angle PQO.$$ 

Thus $OP = OQ$. Since $OC = OR$ as well, we get the conclusion.

Denote by $X$ and $Y$ the feet from $R$ to $CA$ and $CB$, so $\triangle CXR \cong \triangle CYR$. Then, let $t = \frac{CQ}{CR} = 1 - \frac{CP}{CR}$.

Then it follows that

$$[RQL] = [XQL] = t(1 - t) \cdot [XRC] = t(1 - t) \cdot [YCR] = [YP] = [RKP]$$

as needed.

**Remark.** Trigonometric approaches are very possible (and easier to find) as well: both areas work out to be $\frac{1}{8} ab \tan \frac{1}{2} C$. 

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![Diagram of triangle ABC with bisectors and perpendicular bisectors](image-url)
§5 IMO 2007/5, proposed by Kevin Buzzard and Edward Crane, UNK

Let $a$ and $b$ be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

As usual,

$$4ab - 1 \mid (4a^2 - 1)^2 \iff 4ab - 1 \mid (4ab \cdot a - b)^2 \iff 4ab - 1 \mid (a - b)^2.$$ 

Then we use a typical Vieta jumping argument. Define

$$k = \frac{(a - b)^2}{4ab - 1}.$$ 

Note that $k = 0 \iff a = b$. So we will prove that $k > 0$ leads to a contradiction.

Indeed, suppose $(a, b)$ is a minimal solution with $a > b$ (we have $a \neq b$ since $k \neq 0$). By Vieta jumping, $(b, \frac{b^2 + k}{a})$ is also such a solution. But now

$$\frac{b^2 + k}{a} \geq a \implies k \geq a^2 - b^2 
\implies \frac{(a - b)^2}{4ab - 1} \geq a^2 - b^2 
\implies a - b \geq (4ab - 1)(a + b)$$

which is absurd for $a, b \in \mathbb{Z}_{>0}$. (In the last step we divided by $a - b > 0$.)
§6 IMO 2007/6

Let \( n \) be a positive integer. Consider
\[
S = \{(x,y,z) \mid x, y, z \in \{0, 1, \ldots, n\}, \ x + y + z > 0\}
\]
as a set of \((n + 1)^3 - 1\) points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains \(S\) but does not include \((0, 0, 0)\).

The answer is \(3n\). Here are two examples of constructions with \(3n\) planes:

- \(x + y + z = i\) for \(i = 1, \ldots, 3n\).
- \(x = i, y = i, z = i\) for \(i = 1, \ldots, n\).

Suppose for contradiction we have \(N < 3n\) planes. Let them be \(a_i x + b_i y + c_i z + 1 = 0\), for \(i = 1, \ldots, N\). Define the polynomials
\[
A(x, y, z) = \prod_{i=1}^{n} (x - i) \prod_{i=1}^{n} (y - i) \prod_{i=1}^{n} (z - i)
\]
\[
B(x, y, z) = \prod_{i=1}^{N} (a_i x + b_i y + c_i z + 1).
\]

Note that \(A(0, 0, 0) = (-1)^n (n!)^3 \neq 0\) and \(B(0, 0, 0) = 1 \neq 0\), but \(A(x, y, z) = B(x, y, z) = 0\) for any \((x, y, z) \in S\). Also, the coefficient of \(x^n y^n z^n\) in \(A\) is 1, while the coefficient of \(x^n y^n z^n\) in \(B\) is 0.

Now, define
\[
P(x, y, z) \overset{\text{def}}{=} A(x, y, z) - \lambda B(x, y, z).
\]
where \(\lambda = \frac{A(0,0,0)}{B(0,0,0)} = (-1)^n (n!)^3\). We now have that

- \(P(x, y, z) = 0\) for any \(x, y, z \in \{0, 1, \ldots, n\}\).
- But the coefficient of \(x^n y^n z^n\) is 1.

This is a contradiction to Alon’s combinatorial nullstellensatz.