IMO 2000 Solution Notes

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This is a compilation of solutions for the 2000 IMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

2. Let a, b, c be positive real numbers with $abc = 1$. Show that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1.$$

3. Let $n \ge 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C so that $\frac{BC}{AB} = \lambda$.

   Determine all values of λ such that, for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

4. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.

   How many ways are there to put the cards in the three boxes so that the trick works?

5. Does there exist a positive integer n such that n has exactly 2000 distinct prime divisors and n divides $2^n + 1$?

6. Let $\overline{AH_1}$, $\overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute triangle ABC. The incircle ω of triangle ABC touches the sides BC, CA and AB at T_1, T_2 and T_3, respectively. Consider the reflections of the lines H_1H_2, H_2H_3, and H_3H_1 with respect to the lines T_1T_2, T_2T_3, and T_3T_1. Prove that these images form a triangle whose vertices lie on ω.

§1 Solutions to Day 1

§1.1 IMO 2000/1, proposed by Sergei Berlov (RUS)

Available online at https://aops.com/community/p354110.

> **Problem statement**
>
> Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

First, we have $\angle EAB = \angle ACM = \angle BAM$ and similarly $\angle EBA = \angle BDM = \angle ABM$. Consequently, $\overline{AB}$ bisects $\angle EAM$ and $\angle EBM$, and hence $\triangle EAB \cong \triangle MAB$.
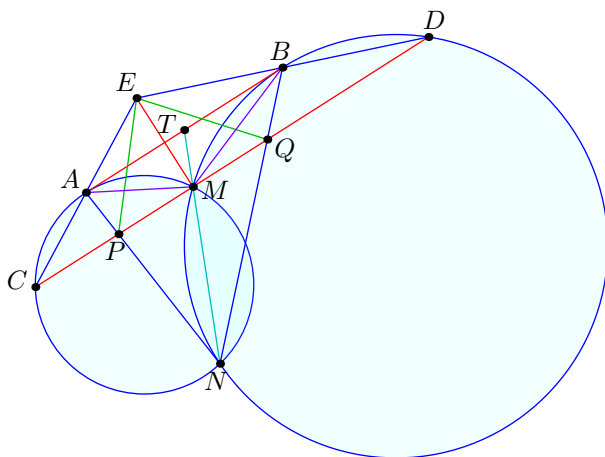
Now it is well-known that $\overline{MN}$ bisects $\overline{AB}$ and since $\overline{AB} \parallel \overline{PQ}$ we deduce that M is the midpoint of $\overline{PQ}$. As $\overline{AB}$ is the perpendicular bisector of $\overline{EM}$, it follows that $EP = EQ$ as well.

§1.2 IMO 2000/2, proposed by Titu Andreescu (USA)

Available online at https://aops.com/community/p354109.

> **Problem statement**
>
> Let a, b, c be positive real numbers with $abc = 1$. Show that
> $$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1.$$

Let $a = x/y$, $b = y/z$, $c = z/x$ for $x, y, z > 0$. Then the inequality rewrites as

$$(-x + y + z)(x - y + z)(x + y - z) \le xyz$$

which when expanded is equivalent to Schur's inequality. Alternatively, if one wants to avoid appealing to Schur, then the following argument works:

- At most one term on the left-hand side is negative; if that occurs we are done from $xyz > 0 > (-x + y + z)(x - y + z)(x + y - z)$.

- If all terms in the left-hand side are nonnegative, let us denote $m = -x + y + z \ge 0$, $n = x - y + z \ge 0$, $p = x + y - z \ge 0$. Then it becomes

$$mnp \le \frac{(m + n)(n + p)(p + m)}{8}$$

which follows by AM-GM.

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§1.3 IMO 2000/3, proposed by Sergei Shikh and Igor Voronovich (BLR)

Available online at https://aops.com/community/p354112.

Problem statement

Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C so that $\frac{BC}{AB} = \lambda$.

Determine all values of λ such that, for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

The answer is $\lambda \geq \frac{1}{n-1}$.

We change the problem by replacing the fleas with **bowling balls** $B_1, B_2, \ldots, B_n$ in that order. Bowling balls aren't exactly great at jumping, so each move can now be described as follows:

- Select two indices $i < j$. Then ball B_i moves to B_{i+1}'s location, B_{i+1} moves to B_{i+2}'s location, and so on; until B_{j-1} moves to B_j's location,

- Finally, B_j moves some distance forward; the distance is at most $\lambda \cdot |B_j B_i|$ and B_j may not pass B_{j+1}.

> **Claim** — If $\lambda < \frac{1}{n-1}$ the bowling balls have bounded movement.

Proof. Let $a_i \geq 0$ denote the initial distance between B_i and B_{i+1}, and let Δ_i denote the distance travelled by ball i. Of course we have $\Delta_1 \leq a_1 + \Delta_2$, $\Delta_2 \leq a_2 + \Delta_3$, ..., $\Delta_{n-1} \leq a_{n-1} + \Delta_n$ by the relative ordering of the bowling balls. Finally, distance covered by B_n is always λ times distance travelled by other bowling balls, so

$$\Delta_n \leq \lambda \sum_{i=1}^{n-1} \Delta_i \leq \lambda \sum_{i=1}^{n-1} ((a_i + a_{i+1} + \cdots + a_{n-1}) + \Delta_n)$$

$$= (n-1)\lambda \cdot \Delta_n + \sum_{i=1}^{n-1} i a_i$$

and since $(n-1)\lambda > 1$, this gives an upper bound. $\qquad \square$

> **Remark.** Equivalently, you can phrase the proof without bowling balls as follows: if $x_1 < \cdots < x_n$ are the positions of the fleas, the quantity
>
> $$L = x_n - \lambda(x_1 + \cdots + x_{n-1})$$
>
> is a monovariant which never increases; i.e. L is bounded above. Since $L > (1 - (n-1)\lambda)x_n$, it follows $\lambda < \frac{1}{n-1}$ is enough to stop the fleas.

> **Claim** — When $\lambda \geq \frac{1}{n-1}$, it suffices to always jump the leftmost flea over the rightmost flea.

Proof. If we let x_i denote the distance travelled by B_1 in the ith step, then $x_i = a_i$ for $1 \leq i \leq n-1$ and $x_i = \lambda(x_{i-1} + x_{i-2} + \cdots + x_{i-(n-1)})$.

In particular, if $\lambda \geq \frac{1}{n-1}$ then each x_i is at least the average of the previous $n-1$ terms. So if the a_i are not all zero, then $\{x_n, \ldots, x_{2n-2}\}$ are all positive and thereafter $x_i \geq \min\{x_n, \ldots, x_{2n-2}\} > 0$ for every $i \geq 2n-1$. So the partial sums of x_i are unbounded, as desired. $\qquad\square$

> **Remark.** Other inductive constructions are possible. Here is the idea of one of them, although the details are more complicated.
>
> We claim in general that given $n-1$ fleas at 0 and one flea at 1, we can get all the fleas arbitrarily close to $\frac{1}{1-(n-1)\lambda}$ (or as far as we want if $\lambda > \frac{1}{n-1}$.). The proof is induction by $n \geq 2$; for $n = 2$ we get a geometric series. For $n \geq 3$, we leave one flea at zero and move the remainder close to $\frac{1}{1-(n-2)\lambda}$, then jump the last flea to $\frac{1+\lambda}{1-(n-2)\lambda}$.
>
> Now we're in the same situation, except we shifted $\frac{1}{1-(n-2)\lambda}$ right and have then scaled everything by $r = \frac{\lambda}{1-(n-2)\lambda}$. If we repeat this process again and check the geometric series, we see the fleas converge to
>
> $$\frac{1}{1-(n-2)\lambda}\left(1 + r + r^2 + r^3 + \ldots\right) = \frac{1}{1-(n-2)\lambda} \cdot \frac{1}{1-r} = \frac{1}{1-(n-1)\lambda}.$$

§2 Solutions to Day 2

§2.1 IMO 2000/4, proposed by Sándor Dobos (HUN)

Available online at https://aops.com/community/p354114.

> **Problem statement**
>
> A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.
>
> How many ways are there to put the cards in the three boxes so that the trick works?

There are $2 \cdot 3! = 12$ ways, which amount to:

- Partitioning the cards modulo 3, or

- Placing 1 alone in a box, 100 alone in a second box, and all remaining cards in the third box.

These are easily checked to work so we prove they are the only ones.

¶ **First solution.**   We proceed by induction on $n \geq 3$ with the base case being immediate.

For the inductive step, consider a working partition of $\{1, 2, \ldots, n\}$. Then either n is in its own box; or deleting n gives a working partition of $\{1, 2, \ldots, n-1\}$. Similarly, either 1 is in its own box; or deleting 1 gives a working partition of $\{2, 3, \ldots, n\}$, and we can reduce all numbers by 1 to get a working partition of $\{1, 2, \ldots, n-1\}$.

Therefore, we only need to consider there cases.

- If 1 and n are both in their own box, this yields one type of solution we already found.

- If n is not in a box by itself, then by induction hypothesis the cards 1 through $n-1$ are either arranged mod 3, or as $\{1\} \cup \{2, 3, \ldots, n-2\} \cup \{n-1\}$.

  - In the former mod 3 situation, since $n + (n-3) = (n-1) + (n-2)$, so n must be in the same box as $n - 3$.

  - In the latter case and for $n > 4$, since $n + 1 = 2 + (n - 1)$, n must be in the same box as 1. But now $n + 2 = (n - 1) + 3$ for $n > 4$, contradiction.

- The case where 1 is in a box by itself is analogous.

This exhausts all cases, completing the proof.

¶ **Second solution.**   Let A, B, C be the sets of cards in the three boxes. Then $A + B$, $B + C$, $C + A$ should be disjoint, and contained in $\{3, 4, \ldots, 199\}$. On the other hand, we have the following famous fact.

> **Lemma**
>
> Let X and Y be finite nonempty sets of real numbers. We have $|X+Y| \geq |X|+|Y|-1$, with equality if and only if X and Y are arithmetic progressions with the same common difference, or one of X and Y is a singleton set.

Putting these two together gives the estimates

$$197 \geq |A+B| + |B+C| + |C+A| \geq 2\left(|A|+|B|+|C|\right) - 3 = 197.$$

So all the inequalities must be sharp. Consequently we conclude that:

> **Claim —** Either the sets A, B, C are disjoint arithmetic progressions with the same common difference $d = \min_{x \neq y \text{ in same set}} |x-y|$, or two of the sets are two singleton. Moreover, $\{3, 4, \ldots, 199\} = (A+B) \sqcup (B+C) \sqcup (C+A)$.

From here it is not hard to deduce the layouts above are the only ones, but there are some details. First, we make the preliminary observation that $3 = 1+2$, $4 = 1+3$, $198 = 98+100$, $199 = 99+100$ and these numbers can't be decomposed in other ways; thus from the remark about the disjoint union:

> **Claim** (Convenient corollary) **—** The pairs $(1,2)$, $(1,3)$, $(98,100)$, $(99,100)$ are all in different sets.

We now consider the four cases.

- If two of the boxes are singletons, it follows from the corollary that we should have $A = \{1\}$, $B = \{100\}$ and $C = \{2, \ldots, 99\}$, up to permutation.

- Otherwise A, B, C are disjoint arithmetic progressions with the same common difference d. As two of $\{1, 2, 3, 4\}$ are in the same box (by pigeonhole), we must have $d \leq 3$.

  - If $d = 3$, then no two elements of different residues modulo 3 can be in the same box, so we must be in the first construction claimed earlier.

  - If $d = 2$, then the convenient corollary tells us we may assume WLOG that $1 \in A$ and $2 \in B$, hence $3 \in C$ (since $3 \notin A$ by convenient corollary, and $3 \notin B$ because common difference 2). Thus we must have $A = \{1\}$, $B = \{2, 4, \ldots, 100\}$ and $C = \{3, 5, \ldots 99\}$ which does not work since $1+4 = 2+3$. Therefore there are no solutions in this case.

  - If $d = 1$, then by convenient corollary the numbers 1 and 2 are in different sets, as are 99 and 100. So we must have $A = \{1\}$, $B = \{2, \ldots, 99\}$, $C = \{100\}$ which we have already seen is valid.

§2.2 IMO 2000/5, proposed by Valerii Senderov (RUS)

Available online at https://aops.com/community/p354115.

> **Problem statement**
>
> Does there exist a positive integer n such that n has exactly 2000 distinct prime divisors and n divides $2^n + 1$?

Answer: Yes.

We say that n is *Korean* if $n \mid 2^n + 1$. First, observe that $n = 9$ is Korean. Now, the problem is solved upon the following claim:

> **Claim** — If $n > 3$ is Korean, there exists a prime p not dividing n such that np is Korean too.

Proof. I claim that one can take any primitive prime divisor p of $2^{2n} - 1$, which exists by Zsigmondy theorem. Obviously $p \neq 2$. Then:

- Since $p \nmid 2^{\varphi(n)} - 1$ it follows then that $p \nmid n$.

- Moreover, $p \mid 2^n + 1$ since $p \nmid 2^n - 1$.

Hence $np \mid 2^n + 1 \mid 2^{np} + 1$ by Chinese Theorem, since $\gcd(n, p) = 1$. $\square$

§2.3 IMO 2000/6, proposed by Lev and Tatiana Emelianovii (RUS)

Available online at https://aops.com/community/p351094.

Problem statement

Let $\overline{AH_1}$, $\overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute triangle ABC. The incircle ω of triangle ABC touches the sides BC, CA and AB at T_1, T_2 and T_3, respectively. Consider the reflections of the lines H_1H_2, H_2H_3, and H_3H_1 with respect to the lines T_1T_2, T_2T_3, and T_3T_1. Prove that these images form a triangle whose vertices lie on ω.

We use complex numbers with ω the unit circle. Let $T_1 = a$, $T_2 = b$, $T_3 = c$. The main content of the problem is to show that the triangle in question has vertices ab/c, bc/a, ca/b (which is evident from a good diagram).
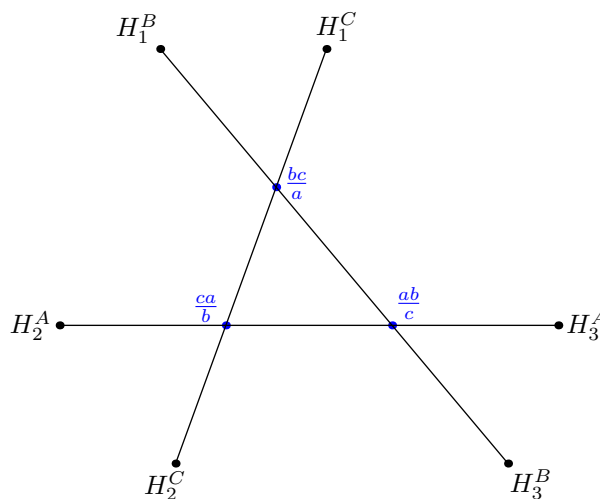
Since $A = \frac{2bc}{b+c}$, we have

$$H_1 = \frac{1}{2}\left(\frac{2bc}{b+c} + a + a - a^2 \cdot \frac{2}{b+c}\right) = \frac{ab + bc + ca - a^2}{b+c}.$$

The reflection of H_1 over $\overline{T_1T_2}$ is

$$H_1^C = a + b - ab\overline{H_1} = a + b - b \cdot \frac{ac + ab + a^2 - bc}{a(b+c)}$$

$$= \frac{a(a+b)(b+c) - b(a^2 + ab + ac - bc)}{a(b+c)} = \frac{c(a^2 + b^2)}{a(b+c)}.$$

Now, we claim that H_1^C lies on the chord joining $\frac{ca}{b}$ and $\frac{cb}{a}$; by symmetry so will H_2^C and this will imply the problem (it means that the desired triangle has vertices ab/c, bc/a, ca/b). A cartoon of this is shown below.

To see this, it suffices to compute

$$H_1^C + \left(\frac{ca}{b}\right)\left(\frac{cb}{a}\right)\overline{H_1^C} = \frac{c(a^2 + b^2)}{a(b+c)} + c^2\frac{\frac{1}{c} \cdot \frac{a^2 + b^2}{a^2 b^2}}{\frac{1}{a}\left(\frac{b+c}{bc}\right)}$$

$$= \frac{c(a^2 + b^2)}{a(b+c)} + \frac{c(a^2 + b^2)}{abc^{-1}(b+c)}$$

$$= \frac{c(a^2 + b^2)}{a(b + c)} \left(\frac{b + c}{b} \right)$$
$$= \frac{c(a^2 + b^2)}{ab} = \frac{ca}{b} + \frac{cb}{a}$$

as desired.