IMO 1999 Solution Notes

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This is an compilation of solutions for the 1999 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. A set S of points from the space will be called completely symmetric if it has at least three elements and fulfills the condition that for every two distinct points A and B from S, the perpendicular bisector plane of the segment AB is a plane of symmetry for S. Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

2. Find the least constant C such that for any integer $n > 1$ the inequality
$$\sum_{1 \le i < j \le n} x_i x_j (x_i^2 + x_j^2) \le C \left(\sum_{1 \le i \le n} x_i \right)^4$$
holds for all real numbers $x_1, \dots, x_n \ge 0$. Determine the cases of equality.

3. Let n be an even positive integer. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

4. Find all the pairs of positive integers (x, p) such that p is a prime and x^{p-1} is a divisor of $(p-1)^x + 1$.

5. Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B Lines MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

6. Find all the functions $f \colon \mathbb{R} \to \mathbb{R}$ such that
$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$
for all $x, y \in \mathbb{R}$.

§1 IMO 1999/1

A set S of points from the space will be called completely symmetric if it has at least three elements and fulfills the condition that for every two distinct points A and B from S, the perpendicular bisector plane of the segment AB is a plane of symmetry for S. Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

Let G be the centroid of S.

Claim — All points of S lie on a sphere Γ centered at G.

Proof. Each perpendicular bisector plane passes through G. So if $A, B \in S$ it follows $GA = GB$. $\square$

Claim — Consider any plane passing through three or more points of S. The points of S in the plane form a regular polygon.

Proof. The cross section is a circle because we are intersecting a plane with sphere Γ. Now if A, B, C are three adjacent points on this circle, by taking the perpendicular bisector we have $AB = BC$. $\square$

If the points of S all lie in a plane, we are done. Otherwise, the points of S determine a polyhedron Π inscribed in Γ. All of the faces of Π are evidently regular polygons, of the same side length s.

Claim — Every face of Π is an equilateral triangle.

Proof. Suppose on the contrary some face $A_1A_2 \dots A_n$ has $n > 3$. Let B be any vertex adjacent to A_1 in Π other than A_2 or A_n. Consider the plane determined by $\triangle A_1A_3B$. This is supposed to be a regular polygon, but arc A_1A_3 is longer than arc A_1B, and by construction there are no points inside these arcs. This is a contradiction. $\square$

Hence, Π has faces all congruent equilateral triangles. This implies it is a regular polyhedron — either a regular tetrahedron, regular octahedron, or regular icosahedron. We can check the regular icosahedron fails by taking two antipodal points as our counterexample. This finishes the problem.

§2 IMO 1999/2

Find the least constant C such that for any integer $n > 1$ the inequality

$$\sum_{1 \le i < j \le n} x_i x_j (x_i^2 + x_j^2) \le C \left(\sum_{1 \le i \le n} x_i \right)^4$$

holds for all real numbers $x_1, \ldots, x_n \ge 0$. Determine the cases of equality.

Answer: $C = \frac{1}{8}$, with equality when two x_i are equal and the remaining x_i are equal to zero.

We present two proofs of the bound.

First solution by smoothing Fix $\sum x_i = 1$. The sum on the left-hand side can be interpreted as $\sum_{i=1}^n x_i^3 \sum_{j \ne i} x_j = \sum_{i=1}^n x_i^3 (1 - x_i)$, so we may rewrite the inequality as: Then it becomes

$$\sum_i (x_i^3 - x_i^4) \le C.$$

> **Claim** (Smoothing) — Let $f(x) = x^3 - x^4$. If $u + v \le \frac{3}{4}$, then $f(u) + f(v) \le f(0) + f(u+v)$.

Proof. Note that

$$\begin{aligned} (u^3 - u^4) + (v^3 - v^4) &\le (u+v)^3 - (u+v)^4 \\ \iff uv(4u^2 + 4v^2 + 6uv) &\le 3uv(u+v) \end{aligned}$$

If $u + v \le \frac{3}{4}$ this is obvious as $4u^2 + 4v^2 + 6uv \le 4(u+v)^2$. □

Observe that if three nonnegative reals have pairwise sums exceeding $\frac{3}{4}$ then they have sum at least $\frac{9}{8}$. Hence we can smooth until $n - 2$ of the terms are zero. Hence it follows

$$C = \max_{a+b=1} (a^3 + b^3 - a^4 - b^4)$$

which is routine computation giving $C = \frac{1}{8}$.

Second solution by AM-GM (Nairit Sarkar) Write

$$\begin{aligned} \text{LHS} &\le \left(\sum_{1 \le k \le n} x_k^2 \right) \left(\sum_{1 \le i < j \le n} x_i x_j \right) = \frac{1}{2} \left(\sum_{1 \le k \le n} x_k^2 \right) \left(\sum_{1 \le i < j \le n} 2x_i x_j \right) \\ &\le \frac{1}{2} \left(\frac{\sum_k x_k^2 + 2\sum_{i<j} x_i x_j}{2} \right)^2 = \frac{1}{8} \left(\sum_{1 \le i < n} x_i \right)^4 \end{aligned}$$

as desired.

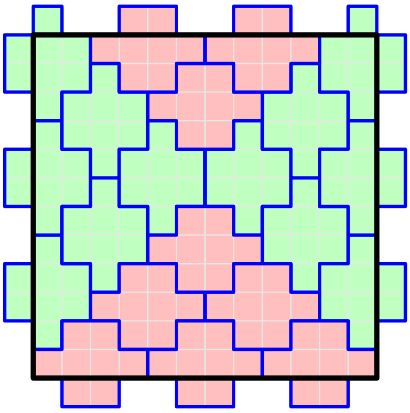
§3 IMO 1999/3

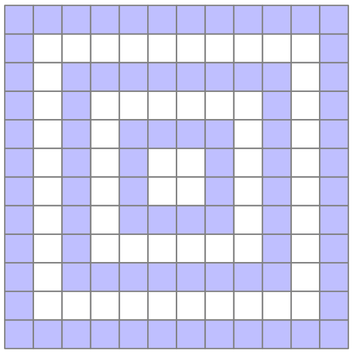
Let n be an even positive integer. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

For every marked cell, consider the marked cell adjacent to it; in this way we have a *domino* of two cells. For each domino, its *aura* consists of all the cells which are adjacent to a cell of the domino. There are up to eight squares in each aura, but some auras could be cut off by the boundary of the board, which means that there could be as few as five squares.

We will prove that $\frac{1}{2}n(n+2)$ is the minimum number of auras needed to cover the board (the auras need not be disjoint).

- A construction is shown on the left below, showing that $\frac{1}{2}n(n+2)$.
- Color the board as shown o the right into "rings". Every aura takes covers exactly (!) four blue cells. Since there are $2n(n+2)$ blue cells, this implies the lower bound.





Note that this proves that a partition into disjoint auras actually always has exactly $\frac{1}{2}n(n+2)$ auras, thus also implying EGMO 2019/2.

§4 IMO 1999/4

Find all the pairs of positive integers (x, p) such that p is a prime and x^{p-1} is a divisor of $(p-1)^x + 1$.

If $p = 2$ then $x \in \{1, 2\}$ and if $p = 3$ then $x \in \{1, 3\}$, since this is IMO 1990/3. We show that there are no other solutions.

Let q be smallest prime divisor of x. We have $q > 2$ since $(p-1)^x + 1$ is odd. Then

$$(p-1)^x \equiv -1 \pmod{q} \implies (p-1)^{2x} \equiv 1 \pmod{q}$$

so the order of $p - 1 \bmod q$ is even and divides $\gcd(q-1, 2x) \le 2$. This means that

$$p - 1 \equiv -1 \pmod{q} \implies p = q.$$

In other words $p \mid x$ and we get $x^{p-1} \mid (p-1)^x + 1$. By exponent lifting lemma, we now have

$$0 < (p-1)\nu_p(x) \le 1 + \nu_p(x).$$

This forces $p = 3$, which we already addressed.

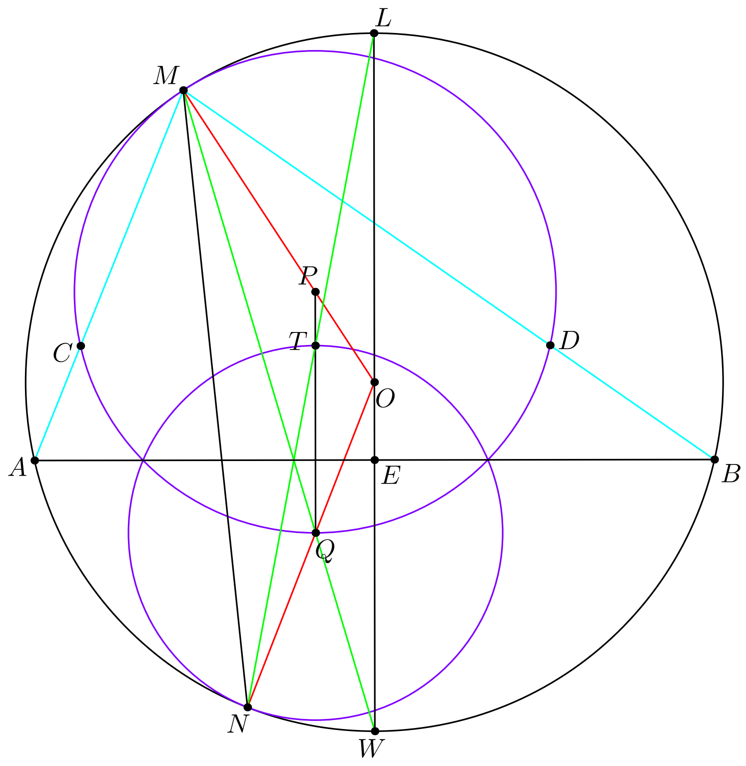
§5 IMO 1999/5

Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B Lines MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

Let P and Q be the centers of Ω_1 and Ω_2.

Let line MQ meet Ω_1 again at W, the homothetic image of Q under $\Omega_1 \to \Omega$.

Meanwhile, let T be the intersection of segment PQ with Ω_2, and let L be its homothetic image on Ω. Since $\overline{PTQ} \perp \overline{AB}$, it follows $\overline{LW}$ is a diameter of Ω. Let O be its center.



Claim — $MNTQ$ is cyclic.

Proof. By Reim: $\measuredangle TQM = \measuredangle LWM = \measuredangle LNM = \measuredangle TNM$. □

Let E be the midpoint of $\overline{AB}$.

Claim — $OEMN$ is cyclic.

Proof. By radical axis, the lines MM, NN, AEB meet at a point R. Then $OEMN$ is on the circle with diameter $\overline{OR}$. □

Claim — MTE are collinear.

Proof. $\measuredangle NMT = \measuredangle TQN = \measuredangle LON = \measuredangle NOE = \measuredangle NME$. □

Now consider the homothety mapping $\triangle WAB$ to $\triangle QCD$. It should map E to a point on line ME which is also on the line through Q perpendicular to $\overline{AB}$; that is, to point T. Hence TCD are collinear, and it's immediate that T is the desired tangency point.

§6 IMO 1999/6

Find all the functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

The answer is $f(x) = -\frac{1}{2}x^2 + 1$ which obviously works.

For the other direction, first note that

$$P(f(y), y) \implies 2f(f(y)) + f(y)^2 - 1 = f(0).$$

We introduce the notation $c = \frac{f(0)-1}{2}$, and $S = \operatorname{img} f$. Then the above assertion says

$$f(s) = -\frac{1}{2}s^2 + (c+1).$$

Thus, the given functional equation can be rewritten as

$$Q(x, s) : f(x - s) = -\frac{1}{2}s^2 + sx + f(x) - c.$$

Claim (Main claim) — We can find a function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x - z) = zx + f(x) + g(z). \qquad (\spadesuit).$$

Proof. If $z \neq 0$, the idea is to fix a nonzero value $s_0 \in S$ (it exists) and then choose x_0 such that $-\frac{1}{2}s_0^2 + s_0x_0 - c = z$. Then, $Q(x_0, s)$ gives an pair (u, v) with $u - v = z$.

But now for any x, using $Q(x + v, u)$ and $Q(x, -v)$ gives

$$\begin{aligned} f(x - z) - f(x) = f(x - u + v) - f(x) &= f(x + v) - f(x) + u(x + v) - \frac{1}{2}u^2 + c \\ &= -vx - \frac{1}{2}s^2 - c + u(x + v) - \frac{1}{2}u^2 + c \\ &= -vx - \frac{1}{2}v^2 + u(x + v) - \frac{1}{2}u^2 = zx + g(z) \end{aligned}$$

where $g(z) = -\frac{1}{2}(u^2 + v^2)$ depends only on z. $\square$

Now, let

$$h(x) := \frac{1}{2}x^2 + f(x) - (2c + 1),$$

so $h(0) = 0$.

Claim — The function h is additive.

Proof. We just need to rewrite $(\spadesuit)$. Letting $x = z$ in $(\spadesuit)$, we find that actually $g(x) = f(0) - x^2 - f(x)$. Using the definition of h now gives

$$h(x - z) = h(x) + h(z). \qquad \square$$

To finish, we need to remember that f, hence h, is known on the image

$$S = \{f(x) \mid x \in \mathbb{R}\} = \left\{ h(x) - \frac{1}{2}x^2 + (2c+1) \mid x \in \mathbb{R} \right\}.$$

Thus, we derive

$$h\left(h(x) - \frac{1}{2}x^2 + (2c+1) \right) = -c \qquad \forall x \in \mathbb{R}. \qquad (\heartsuit)$$

We can take the following two instances of $\heartsuit$:

$$\begin{aligned} h\left(h(2x) - 2x^2 + (2c+1)\right) &= -c \\ h\left(2h(x) - x^2 + 2(2c+1)\right) &= -2c. \end{aligned}$$

Now subtracting these and using $2h(x) = h(2x)$ gives

$$c = h\left(-x^2 - (2c+1)\right).$$

Together with h additive, this implies readily h is constant. That means $c = 0$ and the problem is solved.