IMO 1998 Solution Notes

Compiled by Evan Chen

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This is an compilation of solutions for the 1998 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

2. In a competition, there are a contestants and b judges, where $b \ge 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that for any two judges, their ratings coincide for at most k contestants. Prove that
$$\frac{k}{a} \ge \frac{b-1}{2b}.$$

3. For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that
$$\frac{\tau(n^2)}{\tau(n)} = m.$$

4. Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

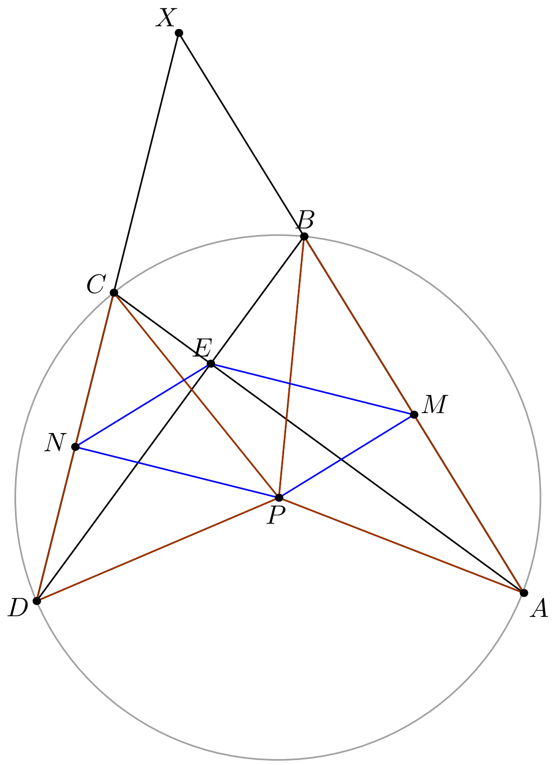
5. Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA, and AB at K, L, and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute.

6. Classify all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the identity
$$f(n^2 f(m)) = m f(n)^2.$$

§1 IMO 1998/1

A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

If $ABCD$ is cyclic, then P is the circumcenter, and $\angle APB + \angle PCD = 180^\circ$. The hard part is the converse.



Let M and N be the midpoints of $\overline{AB}$ and $\overline{CD}$.

Claim — Unconditionally, we have $\measuredangle NEM = \measuredangle MPN$.

Proof. Note that $\overline{EN}$ is the median of right triangle $\triangle ECD$, and similarly for $\overline{EM}$. Hence $\measuredangle NED = \measuredangle EDN = \measuredangle BDC$, while $\measuredangle AEM = \measuredangle ACB$. Since $\measuredangle DEA = 90^\circ$, by looking at quadrilateral $XDEA$ where $X = \overline{CD} \cap \overline{AB}$, we derive that $\measuredangle NED + \measuredangle AEM + \measuredangle DXA = 90^\circ$, so

$$\measuredangle NEM = \measuredangle NED + \measuredangle AEM + 90^\circ = -\measuredangle DXA = -\measuredangle NXM = -\measuredangle NPM$$

as needed. □

However, the area condition in the problem tells us

$$\frac{EN}{EM} = \frac{CN}{CM} = \frac{PM}{PN}.$$

Finally, we have $\angle MEN > 90^\circ$ from the configuration. These properties uniquely determine the point E: it is the reflection of P across line MN.

So $EMPN$ is a parallelogram, and thus $\overline{ME} \perp \overline{CD}$. This implies $\measuredangle BAE = \measuredangle CEM = \measuredangle EDC$ giving $ABCD$ cyclic.

§2 IMO 1998/2

In a competition, there are a contestants and b judges, where $b \ge 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{a} \ge \frac{b-1}{2b}.$$

This is a "routine" problem with global ideas. We count pairs of coinciding ratings, i.e. the number N of tuples

$$(\{J_1, J_2\}, C)$$

of two distinct judges and a contestant for which the judges gave the same rating.

On the one hand, if we count by the judges, we have

$$N \le \binom{b}{2} k$$

by he problem condition.

On the other hand, if $b = 2m + 1$, then each contestant C contributes at least $\binom{m}{2} + \binom{m+1}{2} = m^2$ to N, and so

$$N \ge a \cdot \left(\frac{b-1}{2}\right)^2$$

Putting together the two estimates for N yields the conclusion.

§3 IMO 1998/3

For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that

$$\frac{\tau(n^2)}{\tau(n)} = m.$$

The answer is odd integers m only. If we write $n = p_1^{e_1} \dots p_k^{e_k}$ we get

$$\prod \frac{2e_i + 1}{e_i + 1} = m.$$

It's clear now that m must be odd, since every fraction has odd numerator.

We now endeavor to construct odd numbers. The proof is by induction, in which we are curating sets of fractions of the form $\frac{2e+1}{e+1}$ that multiply to a given target.

The base cases are easy to verify by hand. Generally, assume $p = 2^t k - 1$ is odd, where k is odd. Then we can write

$$\frac{2^{2t}k - 2^t(k+1) + 1}{2^{2t-1}k - 2^{t-1}(k+1) + 1} \cdot \frac{2^{2t-1}k - 2^{t-1}(k+1) + 1}{2^{2t-2}k - 2^{t-2}(k+1) + 1} \cdot \dots \cdot \frac{2^{t+1}k - 2(k+1) + 1}{2^t k - 2^0(k+1) + 1}.$$

Note that $2^{2t}k - 2^t(k+1) + 1 = (2^t k - 1)(2^t - 1)$, and $2^t k - k = k(2^t - 1)$, so the above fraction simplifies to

$$\frac{2^t k - 1}{k}$$

meaning we just need to multiply by k, which we can do using induction hypothesis.

§4 IMO 1998/4

Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

The answer is $(7k^2, 7k)$ for all $k \ge 1$, as well as $(11, 1)$ and $(49, 1)$.

We are given $xy^2 + y + 7 \mid x^2y + x + y$. Multiplying the right-hand side by y gives

$$xy^2 + y + 7 \mid x^2y^2 + xy + y^2$$

Then subtracting x times the left-hand side gives

$$xy^2 + y + 7 \mid y^2 - 7x.$$

We consider cases based on the sign of $y^2 = 7x$.

- If $y^2 > 7x$, then $0 < y^2 - 7x < xy^2 + y + 7$, contradiction.
- If $y^2 = 7x$, let $y = 7k$, so $x = 7k^2$. Plugging this back in to the original equation reads
  $$343k^4 + 7k + 7 \mid 343k^5 + 7k^2 + 7k$$
  which is always valid, hence these are all solutions.
- If $y^2 < 7x$, then $|y^2 - 7x| \le 7x$, so $y \in \{1, 2\}$.

  When $y = 1$ we get
  $$x + 8 \mid x^2 + x + 1 \iff x + 8 \mid 64 - 8 + 1 = 57.$$
  This has solutions $x = 11$ and $x = 49$.

  When $y = 2$
  $$\begin{aligned} 4x + 9 &\mid 2x^2 + x + 2 \\ \implies 4x + 9 &\mid 16x^2 + 8x + 16 \\ \implies 4x + 9 &\mid 81 - 18 + 16 = 79 \end{aligned}$$
  which never occurs.

§5 IMO 1998/5

Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA, and AB at K, L, and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute.

Observe that $\triangle MKL$ is acute with circumcenter I. We now present two proofs.

First simple proof (grobber) The problem is equivalent to showing $BI^2 > BR \cdot BS$. But from

$$\triangle BRK \sim \triangle MKL \sim \triangle BLS$$

we conclude

$$BR = t \cdot \frac{MK}{ML}, \qquad BS = t \cdot \frac{ML}{MK}$$

where $t = BK = BL$ is the length of the tangent from B. Hence $BR \cdot BS = t^2$. Since $BI > t$ is clear, we are done.

Second projective proof Let N be the midpoint of $\overline{KL}$, and let ray MN meet the incircle again at P.

Note that line $\overline{RBS}$ is the polar of N. By Brokard's theorem, lines MK and PL should thus meet the polar of N, so we conclude $R = \overline{MK} \cap \overline{PL}$. Analogously, $S = \overline{ML} \cap \overline{PK}$.

Again by Brokard's theorem, $\triangle NRS$ is self-polar, so N is the orthocenter of $\triangle RIS$. Since N lies between I and B we are done.

§6 IMO 1998/6

Classify all functions $f\colon \mathbb{N} \to \mathbb{N}$ satisfying the identity

$$f(n^2 f(m)) = mf(n)^2.$$

Let $\mathcal{P}$ be the set of primes, and let $g\colon \mathcal{P} \to \mathcal{P}$ be any involution on them. Extend g to a completely multiplicative function on $\mathbb{N}$. Then $f(n) = dg(n)$ is a solution for any $d \in \mathbb{N}$ which is fixed by g.

It's straightforward to check these all work, since $g\colon \mathbb{N} \to \mathbb{N}$ is an involution on them. So we prove these are the only functions.

Let $d = f(1)$.

Claim — We have $df(n) = f(dn)$ and $d \cdot f(ab) = f(a)f(b)$.

Proof. Let $P(m, n)$ denote the assertion in the problem statement. Off the bat,

- $P(1, 1)$ implies $f(d) = d^2$.
- $P(n, 1)$ implies $f(f(n)) = d^2 n$. In particular, f is injective.
- $P(1, n)$ implies $f(dn^2) = f(n)^2$.

Then

$$\begin{aligned} f(a)^2 f(b)^2 &= f(da^2)f(b)^2 && \text{by third bullet} \\ &= f(b^2 f(f(da^2))) && \text{by problem statement} \\ &= f(b^2 \cdot d^2 \cdot da^2) && \text{by second bullet} \\ &= f(dab)^2 && \text{by third bullet} \\ \implies f(a)f(b) &= f(dab). \end{aligned}$$

This implies the first claim by taking $(a, b) = (1, n)$. Then $df(a) = f(da)$, and so we actually have $f(a)f(b) = df(ab)$. □

Claim — All values of f are divisible by d.

Proof. We have

$$\begin{aligned} f(n^2) &= \frac{1}{d} f(n)^2 \\ f(n^3) &= \frac{f(n^2)f(n)}{d} = \frac{f(n)^3}{d^2} \\ f(n^4) &= \frac{f(n^3)f(n)}{d} = \frac{f(n)^4}{d^3} \end{aligned}$$

and so on, which implies the result. □

Then, define $g(n) = f(n)/d$. We conclude that g is completely multiplicative, with $g(1) = 1$. However, $f(f(n)) = d^2 n$ also implies $g(g(n)) = n$, i.e. g is an involution. Moreover, since $f(d) = d^2$, $g(d) = d$.

All that remains is to check that g must map primes to primes to finish the description in the problem. This is immediate; since g is multiplicative and $g(1) = 1$, if $g(g(p)) = p$ then $g(p)$ can have at most one prime factor, hence $g(p)$ is itself prime.

Remark. The IMO problem actually asked for the least value of $f(1998)$. But for instruction purposes, it is probably better to just find all f. Since $1998 = 2 \cdot 3^3 \cdot 37$, this answer is $2^3 \cdot 3 \cdot 5 = 120$, anyways.