Extremely Long and Soggy Math Olympiad

28th ELSMO
Greensboro, NC

Year: **2026**
Day: **1**

June 13, 2026
12:30PM-5:00PM EDT

Problem 1. Geometry is the study of shapes and the properties of them. In geometry, define a *point* to be an axiomatic object in geometry that cannot be easily described. A point may also be referred as a *vertex* (plural: *vertices*) in the following description. A point is usually labeled as a capital symbol such as A. Define a *line* to be an axiomatic object in geometry formed by extending a set of points in a straight fashion beyond in two directions infinitely in both directions. Define a *segment* an object in geometry with two vertices dividing a line that connects them. Define the measure of how long the segment is be the segment's *length*. Segments and lines are usually labeled with two capital symbols like "segment AB" or "line AB". Call the *distance* between two points be the length of the line segment that has vertices being these two points. Moreover, define a *circle* to be the locus of points that are exactly a fixed distance r away from a fixed point. This fixed distance is called the *radius* of the circle, and this fixed point is called the *center* of the circle. Sometimes one labels the name of a circle with a greek symbol such as Ω.The intersection between two mathematical objects are the points that lie in both the objects. A line is *tangent* to a circle if they intersect at exactly one point, or more specifically, they share exactly one common point on their boundaries.

Now, it is possible to introduce the problem after one more definition. In this problem, define a *reflection* as an isometry of Euclidean space fixing a line. Moreover, define the reflection of a point over a line as the point that lies on the opposite side of the line such that the locus of the points that are equidistant (equal distance) to the point and the point's reflection is precisely the entirety of the line.

Let P_1, P_2, and P_3 be three distinct points on the circle Ω. Define P_4 as the reflection of P_1 over line P_2P_3, P_5 as the reflection of P_2 over line P_3P_4, P_6 as the reflection of P_3 over line P_4P_5, P_7 as the reflection of P_4 over line P_5P_6, P_8 as the reflection of P_5 over line P_6P_7, P_9 as the reflection of P_6 over line P_7P_8, P_{10} as the reflection of P_7 over line P_8P_9, P_{11} as the reflection of P_8 over line P_9P_{10}, P_{12} as the reflection of P_9 over line $P_{10}P_{11}$, P_{13} as the reflection of P_{10} over line $P_{11}P_{12}$, P_{14} as the reflection of P_{11} over line $P_{12}P_{13}$, P_{15} as the reflection of P_{12} over line $P_{13}P_{14}$, P_{16} as the reflection of P_{13} over line $P_{14}P_{15}$, P_{17} as the reflection of P_{14} over line $P_{15}P_{16}$, P_{18} as the reflection of P_{15} over line $P_{16}P_{17}$, P_{19} as the reflection of P_{16} over line $P_{17}P_{18}$, P_{20} as the reflection of P_{17} over line $P_{18}P_{19}$, P_{21} as the reflection of P_{18} over line $P_{19}P_{20}$, P_{22} as the reflection of P_{19} over line $P_{20}P_{21}$, P_{23} as the reflection of P_{20} over line $P_{21}P_{22}$, P_{24} as the reflection of P_{21} over line $P_{22}P_{23}$, P_{25} as the reflection of P_{22} over line $P_{23}P_{24}$, and P_{26} as the reflection of P_{23} over line $P_{24}P_{25}$. Prove that line $P_{20}P_{26}$ is tangent to circle Ω.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Problem 2. Gaius Julius Caesar (12 or 13 July 100 BC – 15 March 44 BC) was a Roman general, statesman, and author who was the dictator of the Roman Republic almost continuously from 49 BC until his assassination in 44 BC. A member of the First Triumvirate, he led the Roman armies through the Gallic Wars and defeated his political rival Pompey in Caesar's civil war. He consolidated power and proclaimed himself dictator for life in 44 BC, helping create the political conditions that led to the collapse of the Roman Republic and the emergence of the Roman Empire. For his role in these events, he is regarded as one of history's most influential figures.

In addition to being a great and famous military man, Caesar was a big fan of many other things. For example, he really loved a green salad of romaine lettuce and croutons commonly dressed with lemon juice (or lime juice), olive oil, eggs, Worcestershire sauce, anchovies, garlic, Dijon mustard, Parmesan and black pepper. However, he was also a big fan of geometry. He loved drawing shapes on his wax tablets with his stylus. He liked circles, but he wasn't great at freehanding, so most of them ended up looking like potatoes, even though Caesar didn't know that potatoes existed. He also liked triangles, except that every time he drew one his friend Brutus would start yapping about the Feuerbach hyperbola and the Darboux cubic. Caesar hated these, which made Brutus dislike him very much. At one point, Brutus remarked that if Caesar kept talking about boring triangle centers like the Spieker center then he would stab him.

But most of all he liked squares. He loved square numbers: during Senate meetings, he talked about square numbers so much that people like Brutus got annoyed by him talking about squares all the time and chose a date where both the month and day were 1 less than a square and killed him on that day just so that he couldn't enjoy another squareful date. But most of all he loved square shapes. He loved them so much that he drew smaller squares inside of his squares. One day he was drawing a bunch of small squares inside a big square and he realized he made a grid, where there were n rows of n squares each, all within a bigger square. The picture was really nice and satisfying so Caesar showed it to all the other people in the Senate, but they didn't enjoy geometry as much as Caesar did so they thought it was a crossword and started writing symbols inside the squares. This made Caesar really sad because his beautiful picture was ruined.

However, he noticed something special. He was looking through the grid, and noticed that the first row had exactly one of the symbol "A". Then he saw that the second row also had exactly one of the symbol "A". Then he saw that the third row also had exactly one of the symbol "A". Then he saw that the fourth row also had exactly one of the symbol "A". Then he saw that the fifth row also had exactly one of the symbol "A". He spent the entire senate meeting just looking down the rows of the grid, noticing that every row had exactly one of the symbol "A", and that made some of the other senators pretty fed up and the citizens were pretty mad that their tax money was being spent on Caesar looking through rows noticing that each one had exactly one of the symbol "A".

But then at the next senate meeting he noticed that the first column had exactly one of the symbol "A". Then he noticed that the second column had exactly one of the symbol "A". Then he noticed that the third column had exactly one of the symbol "A". Then he saw that the fourth column had exactly one of the symbol "A". He spent another senate meeting just looking through the columns this time, and to his surprise he found that every column also had exactly one column, which made it extra special because every row and every column both contained exactly one of the symbol "A". He was also very surprised to find a bunch of plebeians outside of his senate hall thing holding lit torches and pitchforks, protesting that their tax money was being spent on Caesar noticing that every column contained exactly one of the symbol "A". He was particularly surprised because pitchforks hadn't been invented yet.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

However, he just executed all of them, and the next day he came in and noticed that the first row had exactly one of the symbol "B". Then he noticed that the second row had exactly one of the symbol "B". Then he noticed that the third row had exactly one of the symbol "B". This happened again for the whole senate meeting, and he noticed that every row and every column of the n by n grid contained exactly one of the symbol "B". However, at the end of that meeting, staring at the fire the protesters had set to his palace, he had a eureka moment and noticed that for every symbol in the grid, each row contained exactly one, and each column also contained exactly one. It was beautiful. Even though he never wrote any of the symbols in the grid, he named it after himself anyway, calling it a "Caesar square". However after he died the Romans decided that calling it that made it sound too much like a salad so they renamed it a "Latin square" because there were many symbols from the Latin alphabet in the square.

On the night of March 14, 44 BC, he brought home his tablet containing the Latin square. While sitting in his burning palace, he was drawing rectangles. Since every square can be taken to a rectangle by a suitable projective transform, Caesar also loved drawing rectangles (but not as much as squares). He started drawing rectangles whose four vertices were four of the cells of the Latin square grid. For the first rectangle he drew, he looked at the four symbols in the four cells that were the vertices of the rectangles. He wrote them down: A, B, C, D. He noticed that these 4 symbols were all different, and he thought that was boring. And then he drew another rectangle and looked at the symbols in the four cells: A, A, B, B. He thought that was cool because there are only two distinct symbols in that list. He stayed up that whole night drawing rectangles in the Latin square, and by eight ante meridian in the morning, he had checked all of the rectangles whose four vertices were cells of the Latin square, and whose sides were aligned with the sides of the Latin square. He had made a great discovery. For every single rectangle he drew, the cells in its vertices contained either two distinct symbols (like A, A, B, B) or four distinct symbols (like A, B, C, D). It was beautiful.

Hurriedly, he ran to the Senate meeting and rushed to tell everyone else. However, the others were bored of his squares and were mad at him for being late so they straight up stabbed him and he died. Then, his palace burned down and all his belongings were burnt down.

In the year 2026, Elmo the historian discovered this story about Caesar and his Latin square. However, he has lost the tablet, so he has no idea what the Latin square itself actually looked like. Help Elmo recover lost details of the story by finding all possible values of n, the side length of the Latin square.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Problem 3. A *pointed set* is a set S, along with a distinguished element $0 \in S$. For example, $\mathbb{Z}$ with 0 as the distinguished element is a pointed set. The set $\mathbb{Z}$, with 1 as the distinguished element instead, is a different pointed set. Pointed sets form the basis for many algebraic structures, such as monoids, groups, rings, and fields. These are obtained by adding additive and multiplicative structure to a pointed set; and the distinguished element becomes the identity element.

Suppose we have pointed sets A, B and a function $f\colon A \to B$ between them. We can define the *image* of f to be the set

$$\operatorname{im} f = \{f(a) : a \in A\}.$$

Note that the image is not neccasarily a *pointed* set, since 0 need not be in the range of f. We additionaly define the *kernel* of f to be the set

$$\ker f = \{a : f(a) = 0\}.$$

Here, 0 refers to the distinguished element of B. Once again, $\ker f$ need not be a pointed set.

Although image and kernel are defined here for functions between pointed sets, these notions admit natural generalizations to many richer algebraic structures, including groups, rings, modules, and vector spaces. In those settings, images and kernels become fundamental tools for studying homomorphisms and understanding how algebraic objects are related. They play a central role in the formulation of quotient structures, the isomorphism theorems, and exact sequences, making them indispensable throughout modern algebra and its applications. We are interested in the last of these applications, which are *exact sequences*.

Consider a sequence of pointed sets A, B, C with functions between them:

$$A \xrightarrow{f} B \xrightarrow{g} C.$$

We say that this sequence is *exact* if $\operatorname{im} f = \ker g$. In other words, the set of elements mapping to the distinguished element 0 of C is the same as the set of elements that f maps to.

Admittedly, this notion of an exact sequence is not that interesting, but once we extend this definition to richer structures like groups, we get a very powerful algebraic idea. Exact sequences provide a concise way to describe how images and kernels of successive maps interact. They are especially important in algebra, where they arise naturally in the study of groups, rings, modules, and vector spaces. Exact sequences encode structural information about algebraic objects and the homomorphisms between them, and they form the language of many advanced topics, including homological algebra, algebraic topology, and algebraic geometry.

For this problem, we will consider the pointed set $\mathbb{Z}/n\mathbb{Z}$, the integers modulo n, where the distinguished element is (the residue class of) 0, and $n \geq 2$ is a positive integer. Any polynomial $P(x) \in (\mathbb{Z}/n\mathbb{Z})[x]$ gives rise to a function $P\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, and thus we can consider the sequence

$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{P(x)} \mathbb{Z}/n\mathbb{Z} \xrightarrow{P(x)} \mathbb{Z}/n\mathbb{Z},$$

where we repeat the function $P(x)$ twice.

Determine all positive integers $n \geq 2$ such that there exists a polynomial $P(x) \in (\mathbb{Z}/n\mathbb{Z})[x]$, *of degree at most* $0.9n$, such that the sequence

$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{P(x)} \mathbb{Z}/n\mathbb{Z} \xrightarrow{P(x)} \mathbb{Z}/n\mathbb{Z}$$

is exact.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

June 18, 2026
12:00PM-4:30PM EDT

Problem 4. Recall the axioms of Zermelo–Fraenkel set theory:

Axiom of extensionality:

$$\forall x \forall y [\forall z (z \in x \iff z \in y) \iff x = y]$$

Axiom of regularity:

$$\forall x [(\exists a (a \in x)) \implies \exists y (y \in x \wedge \neg \exists z (z \in y \wedge z \in x))]$$

Axiom schema of specification:

$$\forall z \forall w_1 \forall w_2 \dots \forall w_n \exists y \forall x [x \in y \iff ((x \in z) \wedge \varphi(x, w_1, w_2, \dots, w_n, z))]$$

denoted as

$$y = \{x \in z \colon \varphi(x)\}$$

Axiom of pairing:

$$\forall x \forall y \exists z \forall a (a \in z \iff a = x \vee a = y)$$

denoted as

$$z = \{x, y\}$$

Axiom of union:

$$\forall \mathcal{F} \exists A \forall Y \forall x [(x \in Y \wedge Y \in \mathcal{F}) \implies x \in A]$$

Axiom schema of replacement:

$$\begin{aligned} &\forall A \forall w_1 \forall w_2 \dots \forall w_n \\ &\qquad [\forall x (x \in A \implies \exists ! y \varphi(x, y, A, w_1, \dots, w_n)) \\ &\qquad \implies \exists B \forall x (x \in A \iff \exists y (y \in B \cap \varphi(x, y, A, w_1, \dots, w_n))] \end{aligned}$$

Axiom of infinity:

$$\exists \omega [\exists \varnothing (\forall z \neg (z \in \varnothing) \wedge \varnothing \in \omega) \wedge \forall y (y \in \omega \implies \exists z (z \in \omega \wedge (a \in z \iff (a \in y \vee a = y))))]$$

Axiom of power set:

$$\forall x \exists y \forall z (\forall q (q \in z \implies q \in x) \iff z \in y)$$

Now, given the above axioms, we can define the class of ordinals $\text{Ord} = \{x \colon \forall y \forall z (y \in z \wedge z \in x \implies y \in x)\}$. Define the successor function $S(A)$ such that $x \in S(A) \iff (x \in A \vee x = A)$. By the axiom of infinity, there exists an ordinal ω such that $\emptyset \in \omega \wedge \forall x (x \in \omega \implies S(x) \in \omega)$. It can be shown that for any two ordinals α and β, either $\alpha = \beta$, $\alpha \in \beta$, or $\beta \in \alpha$. Take ω to be the intersection of all ordinals satisfying the condition in the axiom of infinity. It can be shown that ω is the smallest nonempty ordinal satisfying $\emptyset \in x \wedge \forall x(x \in \omega \implies S(x) \in \omega)$.

Let $0 = \emptyset$. Define addition, multiplication, and exponentiation on the set of natural numbers $\mathbb{N} = \omega$ as follows: For all $a, b \in \mathbb{N}$,

$$\begin{aligned} a + 0 &= a \\ a + S(b) &= S(a+b) \\ a \times 0 &= 0 \\ a \times S(b) &= a \times b + a \\ a^0 &= S(0) \\ a^{S(b)} &= a^b \times a. \end{aligned}$$

From now on, we will abbreviate $a \times b$ as simply ab.

Define the ordered pair $(x, y) = \{\{x, x\}, \{x, y\}\}$. If S and T are sets, we say that the set f is a bijective function from S to T if and only if

$$\begin{aligned} &\forall a(a \in f \implies \exists x \exists y(x \in S \wedge y \in T \wedge a = (x, y))) \\ &\wedge \forall x(x \in S \implies \exists! y(y \in T \wedge (x, y) \in f)) \\ &\wedge \forall y(y \in T \implies \exists! x(x \in T \wedge (x, y) \in f)) \end{aligned}$$

For each $n \in \mathbb{N}$ with $n \neq 0$, define the set $D_n = \{x \in \mathbb{N} \colon \exists y(y \in \mathbb{N} \wedge xy = n)\}$. It can be shown that there exists a unique $\tau(n) \in \mathbb{N}$ with $\tau(n) \neq 0$ and there exists a bijective function from D_n to $\tau(n)$. Define the set $S_n = \{x \in \mathbb{N} \colon \exists i(i \in \mathbb{N} \wedge i \neq 0 \wedge x = \tau(in))\}$.

Prove the following statement:

$$\forall a \forall b(a \in \mathbb{N} \wedge b \in \mathbb{N} \wedge a \neq 0 \wedge b \neq 0 \wedge S_a = S_b \implies \forall n(n \in \mathbb{N} \implies \tau(a^n) = \tau(b^n))).$$

Problem 5. The Norwegian soccer team in Greensboro, North Carolina prepared itself for a legendary battle against the mighty team of France. Having been victorious in 2018 and having lost on penalties in 2022 to Argentina, France was back for revenge, posing a vicious threat to any team near it. Norway was one such team.

The Norwegian coach decided to set a goal for the number of points scored by Norway each game, to be positive. Let $\mathbb{N}$ (for $\mathbb{N}$orway) denote all possible scores that Norway could get in an arbitrarily long soccer game that satisfies this goal.

In order to plan for the vicious French attack on their own goal, however, defenders needed to be assigned to each French player. In particular, France's Kylian Mbappe was very very dangerous. In order to get assistance to creating this assignment, the Norwegian coach phones the local John Berman, who tells him that the set of jerseys of both teams are equal to the set $\mathbb{N}$. Thus the Norwegians need to create an assignment f from the set of French jerseys to the set of Norwegian jerseys. Here due to some Minnesotan physics, the jersey numbers may be arbitrarily long.

Meanwhile, at their training site, the Norwegians practiced shooting the ball along with other drills. They knew that the French possessed the skills to shoot any ball at a $\log x$ trajectory into the goal, making it difficult for the goalie to block the shot. To defeat this, the Norwegians practiced shooting the ball in a trajectory modeled by $\log \log x$.

The Norwegian coaching team had trouble creating a good assignment. Then, Erling Haaland, the star striker of the Norwegian team, comes in. He personally requests that, to match their skill in kicking the ball in an inverse of exponential exponential form, decides that the assignment should follow an exponential exponential format. That is, if we let p be a prime, the mapping f from the set of French jerseys to the set of Norwegian jerseys must satisfy the rule

$$f\left(f(n)^m\right) = p^{f(m)^n}$$

for any jersey numbers m, n. Kevin Yang barges into the room and announces that it is the most convenient for p to be the smallest possible value. The smallest prime is 2. However, the Norwegian coaching team still cannot create an ideal assignment f from the set of French jersey numbers to the set of Norwegian jersey numbers! Help them find all possible assignments f.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Extremely Long and Soggy Math Olympiad

28$^{\text{th}}$ **ELSMO**
GREENSBORO, NC

Year: **2026**

Day: **2**

Problem 6. Let $n \geq 2$ be a positive integer. Some decades ago, while Dennis was in his days of advantage gambling, he went to a casino located in rural Nevada, which offered the game Keno. In the game Keno, every round consists of a machine containing 80 ping-pong balls, which spits out 20 of the ping-pong balls randomly. A player may choose 5 of the 80 ping-pong balls prior to the drawing, and prizes are given out as follows: If 3 of the choices match, the player earns 3 times the bet size. If 4 of the choices match, the player earns 13 times the bet size. If all 5 of the choices match, the player gets the jackpot of 838 times the bet size. It is a straightforward calculation to find that the expected amount of money the player gets back per round is approximately 94 percent of the money put in, so the casino gets back around 6 cents per dollar put in.

Dennis found a casino where the jackpot of 838 times a player's bet size had an added bonus. If he played Keno at the casino enough, the casino would give him an added bonus for the jackpot, increasing the number 838 to 1338. After Dennis found this casino and its promotion for Keno, he calculated the expected return is approximately 1.27 dollars for each dollar put in, much higher than the 94 cents per dollar of the original game. As the advantage gambler that Dennis was at the time, he called the rest of his team of advantage gamblers, a team of $n(n+1)$ people total, and collectively entered this casino to farm off as much money as they possibly can.

This time was already past the turn of the millennium. At this time, casinos had gradually phased out the physical machines that Keno uses and instead used online animations to make the game move faster. Now, instead of waiting for the machine to pop out twenty ping-pong balls, all a player had to do was watch the animated machine on a screen pop out the ping-pong balls much faster than a physical machine could ever do. As a result, Dennis and his team of gamblers were able to farm money off the casino much faster. Dennis and his team went to the casino nine hours a day to mindlessly press buttons every day for two months, each pressing buttons 900 times per hour, earning an average of 367.07 dollars per hour each.

Every time a jackpot was won, gamblers tip the casino workers. Dennis and his team tipped the workers generously throughout their hours on hours at the casino, almost doubling their salary. Hence, although the workers knew exactly what was going on, they had no incentive to tell anyone. Now, most Keno players generally only put a nickel or a dime per round, generally not exceeding a quarter per bet. Dennis and his team, however, were putting one and a half dollars per bet, farming games as much as they could. Due to the large bet sizes that Dennis and his team were putting in, the casino's average bet size increased dramatically. As a large average bet size usually indicates a successful casino, the manager received a bonus for the large bet size. The manager was thus happy with this arrangement, so he had no incentive to stop Dennis and his team either. The only people who would be unhappy with this setup would be the casino chain owners, who happened to be on vacation playing golf for two whole months and had no idea what was going on the whole time.

After the casino chain owners discovered what was going on and shut down the promotion, Dennis and his team decided to gather around a circle and pile their winnings. To speed up the process of sorting their money, Dennis decided to create n groups of consecutive people around the circle for them to sort their money separately first. He did not require the groups to be of equal size, nor that every person be in at least one group. However, every person was required to be part of at most one group. As it turns out, on that day, everyone was wearing a shirt of one of n colors, with an equal number of people wearing each shirt color. Noticing this, Dennis decided to create another rule: Every group of consecutive people must begin and end with two different people with the same shirt color, and this color must be different for all n groups. Prove that there are at least $n+1$ ways to create these groups.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.