Extremely Long & Small[1] Marginalia[2] Olympiad[3]

Year: **2021**

Day: **1**

23rd ELSMO
127.0.0.1[4]

Thursday, June 17, 2021[5]
2:00PM–6:30PM EDT[6]

Problem 1. In $\triangle ABC$, prove that D[7], E[12], X[13], and Y[16] are concyclic.

Problem 2. Given n[19] and A[20], prove that there exists an infinite sequence $b_1, b_2, \ldots$[23] such that S[24] is an integer.

Problem 3. Determine the maximum possible number of diverse cells[25] in $\mathcal{G}$[26].

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[1]referring to the length and size of the marginalia.
[2]referring to the usage of footnotes.
[3]referring to the terms in each problem (points, vocabulary, etc.) being defined by footnotes.
[4]the common address that *localhost* resolves to, indicating the current computer being used.
[5]here we employ the Gregorian calendar system.
[6]also known as 11:00AM–3:30PM PDT, 12:00PM–4:30PM MDT, 1:00PM–5:30PM CDT.
[7]D is the point where the circumcircle of $\triangle AP$[8]Q[9] is tangent to side BC.
[8]P is a point on side AB.
[9]Q is a point on side AC such that the circumcircle of $\triangle AP$[10]Q is tangent to side BC.
[10]P, loc. cit.[11]
[11]short for *loco citato*, meaning "in the place cited."
[12]E lies on side BC such that $BD = EC$.
[13]X is defined as the second intersection of line DP[14] with the circumcircle of $\triangle CDQ$[15].
[14]P, loc. cit.
[15]Q, loc. cit.
[16]Y is defined symmetrically to X; it is the second intersection of line DQ[17] with the circumcircle of $\triangle BDP$[18].
[17]Q, loc. cit.
[18]P, loc. cit.
[19]$n > 1$ is an integer.
[20]here, A is defined as the set $\{a_1, a_2, \ldots, a_n\}$[21].
[21]here, the sequence of integers $a_1, \ldots, a_n$ is defined so that $n \mid a_i - i$[22].
[22]$1 \le i \le n$ is an integer.
[23]where $b_k \in A$ for all positive integers k.
[24]here, S is defined as the sum $\sum_{k=1}^{\infty} \frac{b_k}{n^k}$.
[k]this is an exponent, not a footnote.
[25]a cell is *diverse* if, among the 199 cells in its row or column, every color appears at least once.
[26]$\mathcal{G}$ is a 100×100 grid in which each cell is colored with one of 101 colors.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

Extremely Long & Small Marginalia Olympiad

Year: **2021**

23rd ELSMO
127.0.0.1

Day: **2**

Friday, June 18, 2021
2:00PM–6:30PM EDT

Problem 4. If $\mathbb{N}$[27] $= S_1$[28] $\sqcup$[29] A_1[30], prove that there exists exactly one index i[$2n+27$] such that
$$\frac{1}{d_i}\prod_{j=1}^{n} d_j \in S_i.$$ [$2n+28$]

Problem 5. Up to equivalence[$2n+29$], how many[ω^2] sequences of equi-period[ω^2+1] k are there in the set of sequences with each entry in $[n]$[ω^2+2], in terms of n and k[ω^2+3]?[ω^2+4]

Problem 6. Prove that the inradius of $\triangle ABC$ is twice the inradius of $\triangle DEF$[ω^2+5].

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[27] here, $\mathbb{N}$ refers to the set of positive integers.
[28] here, S_1 is an infinite arithmetic progression.
[29] this symbol refers to the *disjoint union*; that is, we write $A \sqcup B = C$ when $A \cup B = C$ and $A \cap B = \varnothing$.
[30] here, $A_1 = S_2$[31] $\sqcup A_2$[32].
[31] here, S_2 is an infinite arithmetic progression.
[32] here, $A_2 = S_3$[33] $\sqcup A_3$[34].
[33] here, S_3 is an infinite arithmetic progression.
[34] here, $A_3 = S_4$[35] $\sqcup A_4$[36].

$\vdots$

[$2n+24$] here, $A_{n-2} = S_{n-1}$[$2n+25$] $\sqcup S_n$[$2n+26$].
[$2n+25$] here, S_{n-1} is an infinite arithmetic progression.
[$2n+26$] here, S_n is an infinite arithmetic progression.
[$2n+27$] i, loc. cit.
[$2n+28$] for each i, let d_i be the common difference of the arithmetic progression formed by the elements of S_i.
[$2n+29$] two infinite sequences $\{s_i\}_{i\geq 1}$ and $\{t_i\}_{i\geq 1}$ are *equivalent* if the conditions outlined in footnotes i, for $2n+30 \leq i < \omega^2$, all hold.
[$2n+30$] $s_1 = s_1$ if and only if $t_1 = t_1$.
[$2n+31$] $s_1 = s_2$ if and only if $t_1 = t_2$.
[$2n+32$] $s_1 = s_3$ if and only if $t_1 = t_3$.

$\vdots$

[ω] $s_2 = s_1$ if and only if $t_2 = t_1$.
[$\omega+1$] $s_2 = s_2$ if and only if $t_2 = t_2$.
[$\omega+2$] $s_2 = s_3$ if and only if $t_2 = t_3$.

$\vdots$

[$\omega\cdot 2$] $s_3 = s_1$ if and only if $t_3 = t_1$.
[$\omega\cdot 2+1$] $s_3 = s_2$ if and only if $t_3 = t_2$.
[$\omega\cdot 2+2$] $s_3 = s_3$ if and only if $t_3 = t_3$.

$\vdots$

[$\omega(i-1)+(j-1)$] $s_i = s_j$ if and only if $t_i = t_j$.

$\vdots$

[ω^2] in terms of n and k.
[ω^2+1] a sequence $\{r_i\}_{i\geq 1}$ has *equi-period* k if $r_1, r_2, \ldots$ and $r_{k+1}, r_{k+2}, \ldots$ are equivalent.
[ω^2+2] $[n]$ refers to the set $\{1, \ldots, n\}$.
[ω^2+3] n and k are fixed positive integers.
[ω^2+4] that is, determine the largest integer M such that M infinite sequences with equi-period k can be chosen such that no two chosen sequences are equivalent to each other.
[ω^2+5] points D, E, and F lie on sides BC, CA, and AB, respectively, such that quadrilaterals $AFDE$, $BDEF$, and $CEFD$ are tangential[ω^2+6].
[ω^2+6] a convex quadrilateral is *tangential* if there is a circle tangent to all four sides of the quadrilateral.

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