Every *, *, * O*itted

Year: 2020

9th E*O**
127.0.0.1

Day: 1

**onday, Ju*y 20, 2020*
2:00P — 6:30P* EDT*

Prob*e* 1. *et $\mathbb{N}$ be the *et of a** po*itive integer*. Find a** function* $f\colon \mathbb{N} \to \mathbb{N}$ *uch that*

$$f^{f^{f(x)}(y)}(z) = x + y + z + 1$$

for a** $x, y, z \in \mathbb{N}$.

Prob*e* 2. Define the Fibonacci nu*ber* by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. *et k be a po*itive integer. *uppo*e that for every po*itive integer $*$ there exi*t* a po*itive integer n *uch that $* \mid F_n - k$. *u*t k be a Fibonacci nu*ber?

Prob*e* 3. *i**y *e*i**a ha* a device that, when given two di*tinct point* $*$ and $*$ in the p*ane, draw* the perpendicu*ar bi*ector of $**$. *how that if three *ine* for*ing a triang*e are drawn, *e*i**a can *ark the orthocenter of the triang*e u*ing thi* device, a penci*, and no other too**.

Here, $f^a(b)$ denote the re*u*t of a repeated app*ication* of f to b. For*a**y, we define $f^1(b) = f(b)$, and $f^{a+1}(b) = f(f^a(b))$ for a** $a > 0$.

*Ti*e *i*it: 4 hour* 30 *inute*.*
*Each prob*e* i* worth 7 point*.*

Every *, *, * O*itted

Year: 2020

9^{th} **E***O**

127.0.0.1

Day: 2

*Tue*day, Ju*y 21, 2020*
2:00P — 6:30P* EDT*

Prob*e* 4. *et acute *ca*ene triang*e ABC have orthocenter $\ast$ and a*titude AD with D on *ide BC. *et $\ast$ be the *idpoint of *ide BC, and *et D' be the ref*ection of D over $\ast$. *et P be a point on *ine $D'\ast$ *uch that *ine* AP and BC are para**e*, and *et the circu*circ*e* of $\triangle A \ast P$ and $\triangle B \ast C$ *eet again at $\ast \neq \ast$. Prove that $\angle \ast \ast\ast = 90^\circ$.

Prob*e* 5. *et $\ast$ and n be po*itive integer*. Find the **a**e*t po*itive integer $\ast$ for which there exi*t* an $\ast \times n$ rectangu*ar array of po*itive integer* *uch that

- each row contain* n di*tinct con*ecutive integer* in *o*e order,
- each co*u*n contain* $\ast$ di*tinct con*ecutive integer* in *o*e order, and
- each entry i* *e** than or equa* to $\ast$.

Prob*e* 6. For any po*itive integer n, *et

- $\tau(n)$ denote the nu*ber of po*itive integer divi*or* of n,
- $\sigma(n)$ denote the *u* of the po*itive integer divi*or* of n, and
- $\varphi(n)$ denote the nu*ber of po*itive integer* *e** than or equa* to n that are re*ative*y pri*e to n.

et $a, b > 1$ be integer. **a** *a* ha* a ca*cu*ator with three button* that rep*ace the integer n current*y di*p*ayed with $\tau(n)$, $\sigma(n)$, or $\varphi(n)$, re*pective*y. Prove that if the ca*cu*ator current*y di*p*ay* a, then *a* can *ake the ca*cu*ator di*p*ay b after a finite (po**ib*y e*pty) *equence of button pre**e*.

*Ti*e *i*it: 4 hour* 30 *inute*.*
*Each prob*e* i* worth 7 point*.*