abcdEfghijkLMnOpqrtuvwxyz

Year: 2024

26th ELMO

Pittburgh, PA

Day: 1

June 8, 2024
1:20PM-5:50PM EDT

Problem 1. In convex quadrilateral $ABCD$, let diagonal $\overline{AC}$ and $\overline{BD}$ interect at E. Let the circumcircle of ADE and BCE interect $\overline{AB}$ again at $P \neq A$ and $Q \neq B$, repectively. Let the circumcircle of ACP interect $\overline{AD}$ again at $R \neq A$, and let the circumcircle of BDQ interect $\overline{BC}$ again at $\neq B$. Prove that A, B, R, and are concyclic.

Problem 2. For poitive integer a and b, an (a, b)-huffle of a deck of $a + b$ card i any huffle that preerve the relative order of the top a card and the relative order of the bottom b card. Let $n, k, a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$ be fixed poitive integer uch that $a_i + b_i = n$ for all $1 \leq i \leq k$. Big Bird ha a deck of n card and will perform an (a_i, b_i)-huffle for each $1 \leq i \leq k$, in acending order of i. uppoe that Big Bird can revere the order of the deck. Prove that Big Bird can alo achieve any of the $n!$ permutation of the card.

Problem 3. For ome poitive integer n, Elmo write down the equation

$$x_1 + x_2 + \cdots + x_n = x_1 + x_2 + \cdots + x_n.$$

Elmo inert at leat one f to the left ide of the equation and add parenthee to create a valid functional equation. For example, if $n = 3$, Elmo could have created the equation

$$f(x_1 + f(f(x_2) + x_3)) = x_1 + x_2 + x_3.$$

Cookie Monter come up with a function $f : \mathbb{Q} \to \mathbb{Q}$ which i a olution to Elmo' functional equation. (In other word, Elmo' equation i atified for all choice of $x_1, \ldots, x_n \in \mathbb{Q}$). I it poible that there i no integer k (poibly depending on f) uch that $f^k(x) = x$ for all x?

Time limit: 4 hour 30 minute.
Each problem i worth 7 point.

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26^{th} **ELMO**
PITTBURGH, PA

Year: **2024**

Day: **2**

June 15, 2024
1:20PM-5:50PM EDT

Problem 4. Let n be a poitive integer. Find the number of equence $a_0, a_1, a_2, \ldots, a_{2n}$ of integer in the range $[0, n]$ uch that for all integer $0 \le k \le n$ and all nonnegative integer m, there exit an integer $k \le i \le 2k$ uch that $\lfloor k/2^m \rfloor = a_i$.

Problem 5. In triangle ABC with $AB < AC$ and $AB + AC = 2BC$, let M be the midpoint of $\overline{BC}$. Chooe point P on the extenion of $\overline{BA}$ pat A and point Q on egment $\overline{AC}$ uch that M lie on $\overline{PQ}$. Let X be on the oppoite ide of $\overline{AB}$ from C uch that $\overline{AX} \parallel \overline{BC}$ and $AX = AP = AQ$. Let $\overline{BX}$ interect the circumcircle of BMQ again at $Y \neq B$, and let $\overline{CX}$ interect the circumcircle of CMP again at $Z \neq C$. Prove that A, Y, and Z are collinear.

Problem 6. For a prime p, let $\mathbb{F}_p$ denote the integer modulo p, and let $\mathbb{F}_p[x]$ be the et of quartic polynomial with coefficient in $\mathbb{F}_p$. Find all p for which there exit a polynomial $P(x) \in \mathbb{F}_p[x]$ uch that for all integer k, there exit ome integer ℓ uch that $P(\ell) \equiv k \pmod{p}$. (Note that there are $p^3(p-1)$ quartic polynomial in $\mathbb{F}_p$ in total.)

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