Elmo Likes Swapping Math Olympiads

Year: **2025**

27th ELSMO
Aurora, IL

Day: **1**

June 14, 2025
12:30PM-5:00PM CDT

Problem 1. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for all positive integers m and n,

$$f^m(n) + f(mn) = f(m)f(n).$$

Note: $f^m(n) = \underbrace{f(f(\cdots f}_{m \text{ times}}(n)\cdots))$, that is, f applied m times to n.

Problem 2. A contest has 2025 problems. Elmo initially knows that the answers are 1, 2, …, 2025 in some order. Every minute, Elmo selects a problem that has not been answered correctly, inputs an answer to it, and is told whether the answer is correct or incorrect. Elmo wins the contest if and only if he answers all 2025 problems correctly in at most 2024 attempts each. Can Elmo guarantee that he wins the contest?

Problem 3. Let n be a positive integer and p be a prime. In terms of n and p, find the largest nonnegative integer k for which there exists a polynomial $P(x)$ with integer coefficients satisfying the following conditions:

- The x^n coefficient of $P(x)$ is 1.
- p^k divides $P(x)$ for all integers x.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

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Year: 2025

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Day: 2

June 21, 2025
12:30PM-5:00PM CDT

Problem 4. In convex quadrilateral $ABCD$ with $\angle BAD = \angle BCD < 90^\circ$, diagonal AC intersects the circumcircle of $\triangle BCD$ at a point $P \neq C$. Let Q, R, S, and T be the reflections of P across $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Prove that the circumcircle of $\triangle AQT$ is tangent to line RS.

Problem 5. Let k be a positive integer. Define a sequence of positive integers $a_1, a_2, \ldots$ by $a_1 = 1$ and

$$a_{n+1} = a_n + k^{a_n}$$

for all positive integers n. Show that there exist infinitely many primes p such that for any integer r, there exists an index m such that $a_m \equiv r \pmod{p}$.

Problem 6. Let $n \geq 2$ be an integer. An $n \times n$ grid is filled with the numbers 1 through n such that each row contains every number exactly once and any two adjacent rows differ by exactly two elements. Find all n such that for any valid labeling of the grid, there exists a set of n cells, each with a different number, such that no two are on the same row or column.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.