June 8, 2024
1:20PM-5:50PM EDT

Problem 1. In convex quadrilateral $ABCD$, let diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. Let the circumcircles of ADE and BCE intersect $\overline{AB}$ again at $P \neq A$ and $Q \neq B$, respectively. Let the circumcircle of ACP intersect $\overline{AD}$ again at $R \neq A$, and let the circumcircle of BDQ intersect $\overline{BC}$ again at $S \neq B$. Prove that A, B, R, and S are concyclic.

Problem 2. For positive integers a and b, an (a, b)-shuffle of a deck of $a + b$ cards is any shuffle that preserves the relative order of the top a cards and the relative order of the bottom b cards. Let $n, k, a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$ be fixed positive integers such that $a_i + b_i = n$ for all $1 \leq i \leq k$. Big Bird has a deck of n cards and will perform an (a_i, b_i)-shuffle for each $1 \leq i \leq k$, in ascending order of i. Suppose that Big Bird can reverse the order of the deck. Prove that Big Bird can also achieve any of the $n!$ permutations of the cards.

Problem 3. For some positive integer n, Elmo writes down the equation

$$x_1 + x_2 + \cdots + x_n = x_1 + x_2 + \cdots + x_n.$$

Elmo inserts at least one f to the left side of the equation and adds parentheses to create a valid functional equation. For example, if $n = 3$, Elmo could have created the equation

$$f(x_1 + f(f(x_2) + x_3)) = x_1 + x_2 + x_3.$$

Cookie Monster comes up with a function $f : \mathbb{Q} \to \mathbb{Q}$ which is a solution to Elmo's functional equation. (In other words, Elmo's equation is satisfied for all choices of $x_1, \ldots, x_n \in \mathbb{Q}$). Is it possible that there is no integer k (possibly depending on f) such that $f^k(x) = x$ for all x?

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

June 15, 2024
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Problem 4. Let n be a positive integer. Find the number of sequences $a_0, a_1, a_2, \ldots, a_{2n}$ of integers in the range $[0, n]$ such that for all integers $0 \leq k \leq n$ and all nonnegative integers m, there exists an integer $k \leq i \leq 2k$ such that $\lfloor k/2^m \rfloor = a_i$.

Problem 5. In triangle ABC with $AB < AC$ and $AB + AC = 2BC$, let M be the midpoint of $\overline{BC}$. Choose point P on the extension of $\overline{BA}$ past A and point Q on segment $\overline{AC}$ such that M lies on $\overline{PQ}$. Let X be on the opposite side of $\overline{AB}$ from C such that $\overline{AX} \parallel \overline{BC}$ and $AX = AP = AQ$. Let $\overline{BX}$ intersect the circumcircle of BMQ again at $Y \neq B$, and let $\overline{CX}$ intersect the circumcircle of CMP again at $Z \neq C$. Prove that A, Y, and Z are collinear.

Problem 6. For a prime p, let $\mathbb{F}_p$ denote the integers modulo p, and let $\mathbb{F}_p^{(4)}[x]$ be the set of quartic polynomials with coefficients in $\mathbb{F}_p$. Find all p for which there exists a polynomial $P(x) \in \mathbb{F}_p^{(4)}[x]$ such that for all integers k, there exists some integer ℓ such that $P(\ell) \equiv k \pmod{p}$. (Note that there are $p^4(p-1)$ quartic polynomials in $\mathbb{F}_p$ in total.)

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.