ELMOnade

23rd ELMO
PITTSBURGH, PA

Year: **2023**

Day: **1**

Saturday, June 10, 2023
1:20PM–5:50PM EDT

Problem 1. Let m be a positive integer. Find, in terms of m, all polynomials $P(x)$ with integer coefficients such that for every integer n, there exists an integer k such that $P(k) = n^m$.

Problem 2. Let a, b, and n be positive integers. A lemonade stand owns n cups, all of which are initially empty. The lemonade stand has a *filling machine* and an *emptying machine*, which operate according to the following rules:

- If at any moment, a completely empty cups are available, the filling machine spends the next a minutes filling those a cups simultaneously and doing nothing else.
- If at any moment, b completely full cups are available, the emptying machine spends the next b minutes emptying those b cups simultaneously and doing nothing else.

Suppose that after a sufficiently long time has passed, both the filling machine and emptying machine work without pausing. Find, in terms of a and b, the least possible value of n.

Problem 3. Convex quadrilaterals $ABCD$, $A_1B_1C_1D_1$, and $A_2B_2C_2D_2$ are similar with vertices in order. Points A, A_1, B_2, B are collinear in order, points B, B_1, C_2, C are collinear in order, points C, C_1, D_2, D are collinear in order, and points D, D_1, A_2, A are collinear in order. Diagonals AC and BD intersect at P, diagonals A_1C_1 and B_1D_1 intersect at P_1, and diagonals A_2C_2 and B_2D_2 intersect at P_2. Prove that points P, P_1, and P_2 are collinear.

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.

waterELMOn

23^{rd} ELMO
Pittsburgh, PA

Year: **2023**

Day: **2**

Sunday, June 18, 2023
1:20PM–5:50PM EDT

Problem 4. Let ABC be an acute scalene triangle with orthocenter H. Line BH intersects $\overline{AC}$ at E and line CH intersects $\overline{AB}$ at F. Let X be the foot of the perpendicular from H to the line through A parallel to $\overline{EF}$. Point B_1 lies on line XF such that $\overline{BB_1}$ is parallel to $\overline{AC}$, and point C_1 lies on line XE such that $\overline{CC_1}$ is parallel to $\overline{AB}$. Prove that points B, C, B_1, C_1 are concyclic.

Problem 5. Find the least positive integer M for which there exist a positive integer n and polynomials $P_1(x)$, $P_2(x)$, …, $P_n(x)$ with integer coefficients satisfying

$$Mx = P_1(x)^3 + P_2(x)^3 + \cdots + P_n(x)^3.$$

Problem 6. For a set S of positive integers and a positive integer n, consider the game of (n, S)-*nim*, which is as follows. A pile starts with n watermelons. Two players, Deric and Erek, alternate turns eating watermelons from the pile, with Deric going first. On any turn, the number of watermelons eaten must be an element of S. The last player to move wins. Let $f(S)$ denote the set of positive integers n for which Deric has a winning strategy in (n, S)-nim.

Let T be a set of positive integers. Must the sequence

$$T,\ f(T),\ f(f(T)),\ \ldots$$

be eventually constant?

Time limit: 4 hours 30 minutes.
Each problem is worth 7 points.